Universidad Loyola Andalucía
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Distributed observers for LTI systems

An approach based on subspace decomposition

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To my family

Abstract

When considering large-scale plants, such as factories, water irrigation channels or solar fields, the problem of state estimation is more difficult to solve than in small-scale systems. It should be noted that information from these systems is frequently collected by many individual agents widespread across geographically remote locations, which complicates estimators' designs. Furthermore, these agents are required to communicate with others to achieve system-wide goals, triggering problems derived from the network topology and communication drawbacks such as delays, quantization, limited bandwidth, etc. This thesis aims to provide new solutions for the problem of distributed estimation of the state of a linear time-invariant (LTI) plant with a network of agents. To achieve this goal, several novel structures for agent-based estimators are presented, based on an orthogonal decomposition of the local observable/unobservable subspaces of each agent.

First, a novel observer is introduced based on a structure that incorporates consensus among the agents and that can be designed in a distributed fashion, achieving a robust solution with good estimation performance. Furthermore, the structure includes the ability to set the convergence rate of the estimator arbitrarily.

Concerning perturbed models, an LQ-based design method for the observer structure is presented, stating stability and optimality conditions and showing in simulation the performance of the algorithm for the unperturbed and perturbed scenarios. The design method presented allows the user, through the use of one scalar parameter, to modify the observer according to their experience with the plant.

Finally, a second observer structure is presented based on the same principle of subspace decomposition, but this time, the scenario is a little different. Each of the agents involved in the network must perform real-time monitoring of the plant's state, counting on local measurements of the state taken by the agents and measurements taken by the rest of the network. This interagent communication takes place within a multihop network. Therefore, the transmitted information suffers delay depending on the position of the sender and receiver in a communication graph. A novel data-fusion-based observer structure is presented, and two main subproblems are addressed: the observer design for stabilizing the estimation error and an optimal observer design to minimize the estimation uncertainties when plant disturbances and measurement noise come into play.

All contributions of this thesis are theoretical in nature. However, the solutions adopted could be applied to a wide variety of distributed systems.

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List of Acronyms

DKF	**D**istributed **K**alman **F**ilter
LMI	**L**inear **M**atrix **I**nequality
LQR	**L**inear **Q**uadratic **R**egulator
LTI	**L**inear **T**ime **I**nvariant
NCS	**N**etworked **C**ontrol **S**ystems
SM	**S**et-**M**embership
CPS	**C**yber-**P**hysical **S**ystem
CPSoS	**C**yber-**P**hysical **S**ystem **o**f **S**ystems
GUUB	**G**lobally **U**niformly **U**ltimately **B**ounded
SoS	**S**ystem **o**f **S**ystems

Chapter 1

Introduction and Objectives

1.1 Introduction

This thesis is devoted to the research and development of new algorithms for the problem of distributed estimation of the state of a plant by a network of agents. The focus of the thesis is not only on the distributed implementation of the estimation algorithms but also on the distributed design of the observer.

The approaches presented in the thesis take into account different reasoning in terms of the presence of noise and perturbations. Under those scenarios, the methods proposed intend to minimize the uncertainties in the estimation.

1.2 Thesis conceptual framework and motivation

This thesis lays at the intersection of NCS, distributed systems and CPSoS. A summary of these three concepts together with their limitations and main challenges are introduced in the following points.

1.2.1 Networked Control Systems

Traditionally, classical control (or estimation) systems consisted of a controller (or estimator) that collected all the measurements of a process and carried out computations to control (or estimate) a plant (see Figure 1.1). To accomplish this goal, it was of crucial importance to place the controller close to the system of interest, located near the sensors and actuators. Another alternative was to introduce point-to-point communication dedicated exclusively to the control loop. Similarly, it was considered a perfect communication channel approach with the following features:

- Constant sampling time;
- Absence of transmission delays and dropouts; and
- Unlimited bandwidth and resolution in the signals.

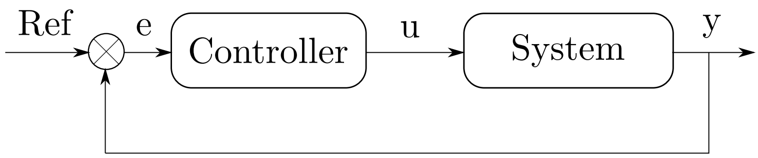

Figure 1.1: Classical control scheme.

These assumptions facilitated the development of control and estimation strategies implemented in the controller or estimator. However, typical centralized systems emerged in the industry and engineering applications.

In contrast with the aforementioned framework, the concept of "Networked Control Systems" arised at the end of the last century. NCSs are spatially distributed systems for which the communication between sensors, actuators and controllers is supported by a shared communication network, as shown in Figure 1.2 [29]. The communication channel comprises a variety of technological solutions such as Bluetooth, ZigBee or Wi-Fi, to mention some of them [39].

The use of a multipurpose shared network to connect spatially distributed elements results in flexible architectures and generally reduces both installation and maintenance costs. Consequently, in the last decades, NCSs have been finding applications in a broad range of areas such as mobile sensors [51], remote surgery [43], haptics collaboration over the Internet [28], automated highway systems and unmanned aerial vehicles [72] (for a more detailed information reader is referred to [29]). However, new challenges are introduced to the scientific community. The occurrence of failures in communications and the devices, lack of instantaneous access to the information of the plant by the controllers and similar issues present a problem for control and estimation design which must be studied.

1.2.2 Distributed systems

The decrease in the production costs of electronic devices has led to the evolution of control systems to distributed architectures in which there are several intelligent "agents" that exchange information with the rest of the agents in the network to control and estimate the state of a system.

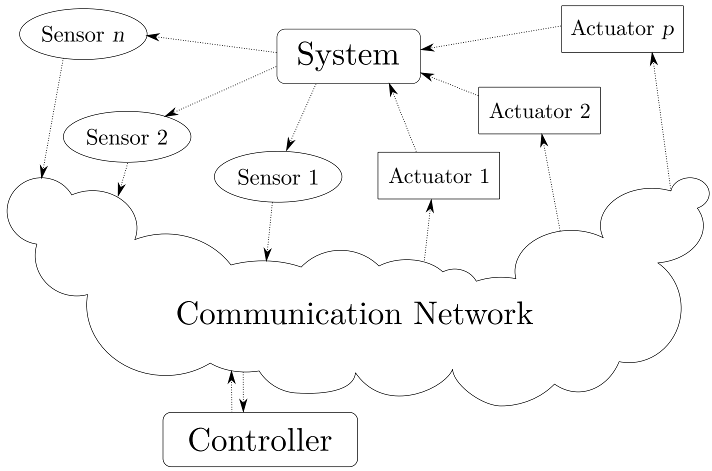

Figure 1.2: Networked Control Systems Scheme.

Thus, a distributed system can be defined as a collection of independent agents that appears to the users of the system as a single device. In order to illustrate this concept, it is probably more helpful to give several examples of distributed systems [75]. Let us consider a network of workstations in a company where, in addition to each user's personal workstation, there might be a pool of processors in the machine room that are not assigned to specific users but are allocated dynamically as needed. Another interesting example is a factory full of robots. Consider that one robot is in charge of the assembly line and it detects that a part it is supposed to install is defective. Then, it communicates with another robot in the parts department to bring a replacement. This is a distributed system.

In this new paradigm, the different agents are geographically sparse and communicate through a communication network (Figure 1.3). Thus, the tasks of monitoring, estimating and controlling the plant are no longer carried out by a single device, as in the centralized case, but require the coordination of all the agents involved in the network.

This architecture has many advantages with respect to the traditional centralized schemes [70]:

- Scalability: sometimes, the plant to be controlled has large dimensions, which make it difficult to implement a centralized structure since the controller must receive signals from very distant locations. Distributed architecture offers the possibility of introducing a network of easily expandable sensors.

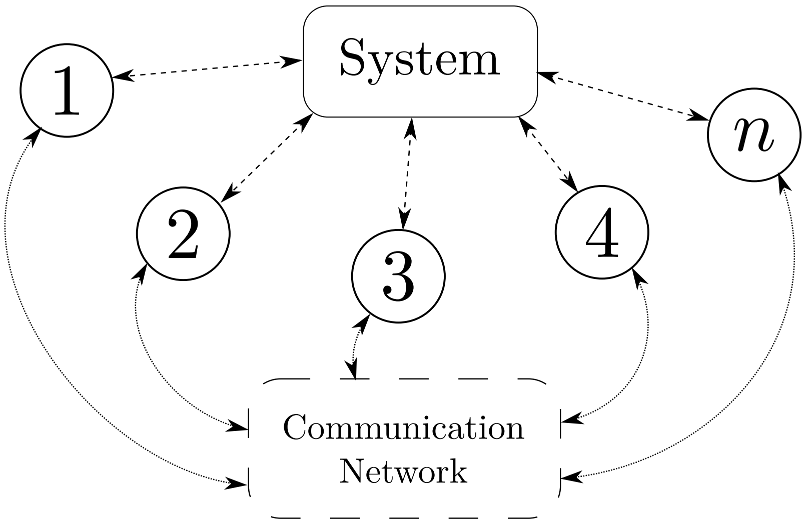

Figure 1.3: Distributed control scheme.

- Flexibility: distributed architecture adapts to any type of plant, allowing the interconnection of different intelligent devices that share a common communication protocol.
- Fault tolerance: while in centralized control schemes the failure of the main controller supposes the loss of control over the plant, in distributed control systems, the loss of a device in the network may not suppose a determining failure and can partially operate the plant in the absence of such a device.
- Machine learning: distributed networks are particularly suitable for learning from large datasets, as they take advantage of the potential collaboration between different devices.

However, although distributed systems have the aforementioned strengths, they also have their weaknesses. One of the main potential problems is related to the communication channel. It can present problems such as failures in communications, delays in the sending of packets, etc. Another problem is caused by the lack of global information in all the devices involved in the problem. Each device in a distributed system has local information about a process but, in contrast with centralized schemes, there is some global information that is not accessible by all of the devices. Finally, the implementation of distributed algorithms is a challenge that has been studied in the last decades and it is still a research focus.

1.2.3 Cyberphysical Systems of Systems (CPSoS)

A SoS is an integration of a finite number of constituent systems. The elements of a SoS are independent, operable and networked together for a period of time to achieve a certain higher goal [30]. Thus, in an SoS, there is a group of systems where most of the components have some managerial and operational independence, but the purpose of the system is to provide a function or service that cannot be provided by the individual systems independently or cannot be provided in an as efficient manner as with the overall system [19].

On the other hand, CPS are physical and engineered systems whose operations are monitored, coordinated, controlled and integrated by a computing and communication core [63]. In [19], CPS are defined as large complex physical systems that interact with a considerable number of distributed computing elements for monitoring, control and management, which can exchange information among themselves and with human users.

Thus, CPSoS are cyberphysical systems that exhibit the features of systems of systems. That is, they are large, are spatially distributed, have complex dynamics and partial autonomy in their subsystems among other features.

The concept of CPSoS has emerged as an active domain of research in recent years in the light of various disciplines, such as computer science, systems control and systems engineering. This new paradigm is of crucial importance for solving societal challenges around the world. For this reason, the European Commission has supported this line of research under the FP7 program [20]. It aims to build constituencies for a European R&I agenda on SoS. Therefore, CPSoS provide a forum and an exchange platform for systems of systems-related communities and ongoing projects, focusing on the challenges posed by the engineering and the operation of technical systems in which computing and communication systems interact with large complex physical systems.

The complexity of all the subsystems considered, together with the communication problems of a real network and the coordination of the systems studied, increase the challenges and difficulties in the development of CPSoS.

1.3 Applications

The last three sections have introduced three different paradigms where the problem of distributed estimation becomes of crucial importance. This section describes some systems at the intersection of these three paradigms where the distributed estimation algorithms can be applied. These applications are enumerated next:

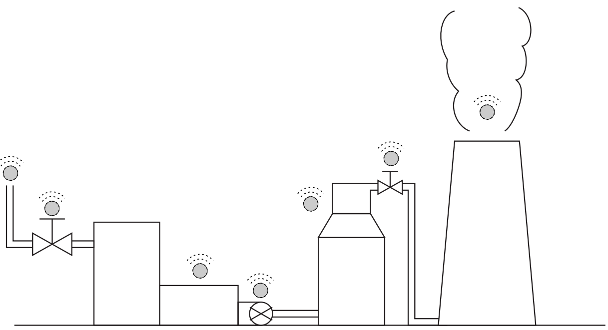

Figure 1.4: Example of a set of agents sparsely deployed in an industrial plant.

- **Control of water distribution networks:** the water distribution network is often composed of thousands of nodes and pipes and it has a length of hundreds of kilometers. This is the main reason for management that is incapable of preventing water leakage and planning optimization actions [4]. Many approaches can be found in the literature dealing with this problem (see for instance [50, 41]) and most of them propose a distributed approach for the problem. In these distributed approaches, the geographically sparse sensors and actuators communicate among them and make decisions about the control actions to accomplish in the plant. To do that, a reconstruction of the state of the system is needed in some of the agents, being crucial the correct implementation of a distributed estimation algorithm.
- **Formation control of autonomous vehicles:** in the maturing field of mobile robot control, a natural extension to the traditional trajectory tracking problem is that of coordinated tracking in its most general formulation. The challenge which that problem addresses consists in finding a coordinated control scheme for multiple robots that make them maintain some given, possibly time-varying, formation at the same time that the robots, viewed as a group, execute a given task [17]. This problem needs the implementation of distributed estimation and control algorithms to prevent coordination problems [65, 22].
- **Transportation and logistics:** transport is a CPS of systems that depends on multiple factors, including the pattern of human settlements, the organization of production and the availability of infrastructure [77]. In this framework, much research and numerous initiatives are being carried out [83, 69]. The interaction between transport, the human necessities and traffic regulators will be of extreme relevance in the future. This approach will also

entail the problem of distributed estimation appearing in a multitude of applications. For instance, consider [68], where a novel CPS framework for aircraft and airspace system design and performance assurance is presented. Another interesting article is [6], which summarizes the results of the research carried out in the international competition, DARPA, in November 2007 for the promotion of research and development on autonomously driving vehicles in urban environments.

- **Electric power grids:** to ensure reliable and quality power supply for all consumers distributed throughout the grid, the operational goals of the grid relate first to maintaining grid stability while adhering to the grid codes, i.e., the network specifications for the operation of the grid, such as voltage level references at different transmission (high-voltage, HV) and distribution (low-voltage, LV) lines, power transfer levels for transmission and distribution and frequency references in the system, the provision of a connection to the grid, the performance of electricity transmission across the grid, and cross-border transmission [77]. One of the main problems arises with the inclusion of distributed renewable generators in low-voltage distribution grids that creates important local imbalances between generation and consumption. The distribution system operators in current practice acts conservatively and prohibit the connections of units with production capacity at certain grid points in situations when worst-case static simulations show that grid code violations might occur. These facts are forcing the evolution of classical electric power grid schemes to distributed architecture with a liberalized market in which each bus or agent is capable of making real-time decisions according to economic and technical factors. In this framework, introducing a well-distributed estimation algorithm will be the key to guarantee the stability of the overall power system. Several challenges in this framework where defined in [74].

- **Smart buildings:** smart grids and smart buildings are related scenarios. In smart buildings, several technologies are combined together to establish reliable and sustainable technology. Thus, this technology deploys green and zero-energy buildings that use all available sources of energy efficiently. In that way, there exist several intelligent devices that control renewable resources, electrical energy storage, combined heat and power plants, etc. These elements work together to optimize the efficiency of the overall system [82].

1.4 Problem statement

The problem of distributed estimation addressed in this thesis is described as follows: consider a group of agents interconnected in a communication network. Each of the agents has access to some plant outputs. The objective of each of them is to reconstruct the complete state of the system in real-time based on the information obtained locally through measurements and the information exchanged within their neighborhood, that is, with those agents with whom they have a channel of direct communication. A schematic of this problem can be found in Figure 1.5.

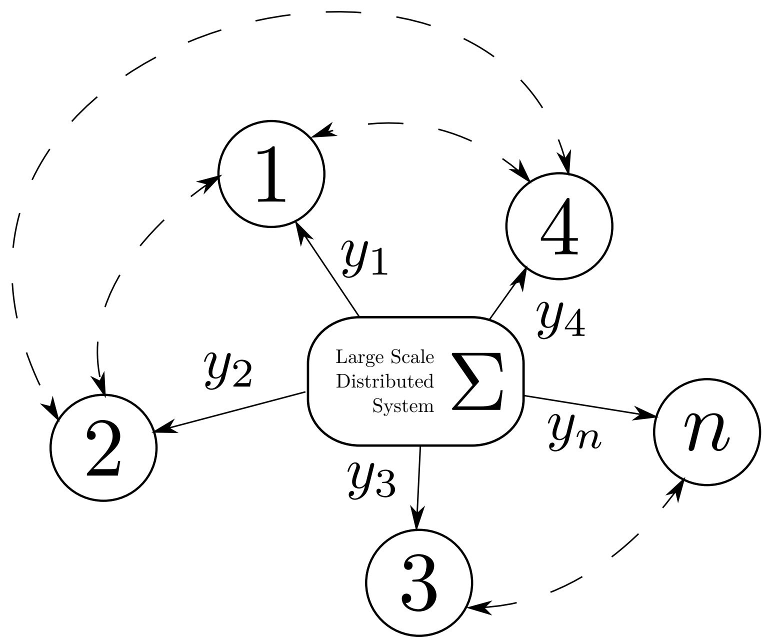

Figure 1.5: Distributed estimation problem scheme, where a set of agents take measurements of system Σ and exchange information among them through a communication network (in dashed lines).

The problem to solve is not only estimating the state of the system operating in a distributed way but also that the design of the estimators must be carried out in a distributed fashion, adding to the algorithm properties of flexibility and scalability.

This problem may present additional hurdles according to given considerations, as it will be seen in the subsequent chapters of the thesis. We consider the following assumptions:

- The system dynamics are known and are modeled as a discrete-time linear time-invariant system.
- The system dynamics and the measurements taken by every agent can be unperturbed, affected by upper-bounded norm perturbations or affected by Gaussian noises.
- The communication network can be modeled as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of agents and $\mathcal{E}$ is the set of links or edges. We consider a robust communication network in which the information exchange between agents is not delayed or lost.
- The system is assumed to be collectively detectable by the network of agents. That is, every agent is able to detect the entire system with the information collected by local measurement of the state and that provided by the network.

It is important to point out that the implementation of a distributed estimation solution consists of three main steps that are described next (an overview of this process is depicted in Figure 1.6):

- First, an exchange of information must be accomplished which allows for the different agents to collect the needed information to design the observer.
- Second, the design of the observer must be carried out. Depending on the observer structure and the solution adopted, this phase can be repeated at each sample time. This design must guarantee the stability in the estimation and, depending on the case, achieve good performance in the presence of noises and disturbances.
- Once the observers have been designed, the running phase is started, in which the observers work toward reconstructing the state by taking measurements and exchanging the adequate information with their neighbors.

All the solution presented in the following chapters relays in a subspace decomposition named "Observability Staircase Form" [67]. This form allows to each agent to identify its observable and unobservable subspace.

1.5 Related literature

Concerning distributed estimation, the literature is broad, and many approaches can be found. Perhaps the most well-known approach to this problem is the

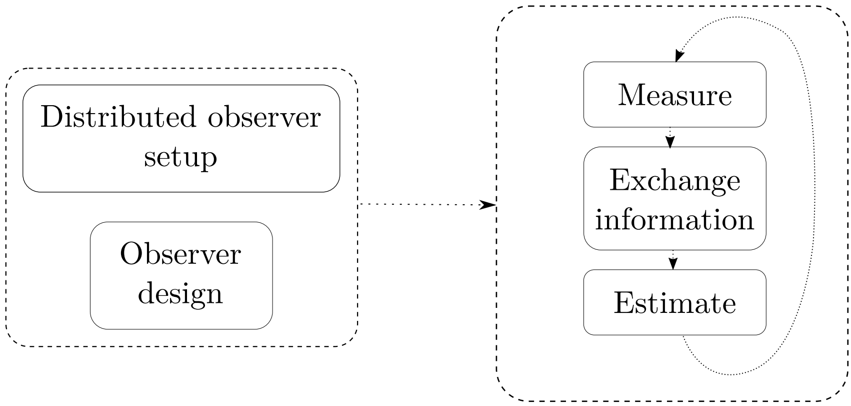

Figure 1.6: Distributed estimation procedure.

DKF, presented for the first time in [52]. The DKF is a distributed estimation strategy that is applicable when the evolution of the system dynamics and the measurements taken by the agents are affected by Gaussian noise. Thus, based on the measurements taken by each agent and the information exchanged with their neighbors, the DKF allows for every agent to minimize the uncertainties in the estimation. Recall that since there are several agents, it is possible that two or more agents measure the same state, which results in a more correct estimation when the measurements are affected by disturbances. This estimation strategy has attracted the attention of many researchers, and numerous approaches can be found in the literature. For example, in [8], the DKF was combined with consensus algorithms to achieve better performance. A different approach can be found in [5], where the DKF was extended to nonlinear systems. Other methods consider the introduction of weights based on the covariance matrices of the estimation error (see, for instance, [31]). On the other hand, the application of diffusion strategies for the distributed Kalman filter has also been studied; a relevant article in this area is [10]. Additionally, there are works that study the convergence time of the distributed Kalman filter with respect to the centralized case [76]. It is worth pointing out that when introducing a communication network in the distributed case, the amount of information exchanged between agents is high, and therefore, its complexity setting a fast convergence rate is also high. A thorough examination of this issue can be found in [76]. Finally, a new approximation based on the decomposition of the state vector in the observable/unobservable subspaces of each agent is presented in [14]. This idea has been extended and developed in [66], where an estimation algorithm based on data fusion is presented. This last paper corresponds to Chapter 4 of the present thesis.

The DKF has many positive features, among which two stand out: it minimizes the uncertainty in estimation when the system and measurements are affected by noise, and it has the ability to be designed in a completely distributed way. This second feature is crucial since it gives flexibility and scalability to the distributed system. DKF techniques provide optimal solutions under the assumption of Gaussian perturbations. However, this assumption is not always reasonable when considering real systems.

Following the same motivation as that for this thesis, new approaches have come out in recent years, which present an observation structure with the capability of being designed in a distributed manner. Most of these new approaches consider noise-free systems and measurements. In that way, the problem is addressed by focusing on finding a design strategy that guarantees stability in estimation, minimizing the exchange of information between agents. Three relevant articles on this subject are [59], [60] and [79]. These three approaches base their strategy on state augmentation. Although they are valid methods, state augmentation increases the dimension of the state, which has repercussions for computational costs of the algorithm and for the exchange of information. Another relevant work in this field is [36], where the authors make use of an orthogonal transformation of the state space in the observable and unobservable subspace for each agent. This technique is similar to that used in the observers presented in this thesis. However, as will be seen in subsequent chapters, the strategy presented here has additional positive aspects. Furthermore, in [36], agents need global information about the communications graph, and the observer design does not allow one to set an arbitrary convergence rate. Another alternative to this same problem can be found in [25], where the distributed observer design is made through the resolution of a LMI problem. This design allows to set a prefixed convergence rate for the estimator. Finally, in [48], the authors present a distributed observer based on the decomposition of the state space in orthogonal subspaces. Although the observer design is innovative, it has some shortcomings, similar to those of similar strategies, such as the lack of possibility to set an arbitrary convergence rate and the large amount of information that needs to be exchanged between agents under moderate assumptions.

A different approach is the set-membership estimation [44]. This approach is based on the construction of a compact set, that includes, with guarantee, the states of the systems that are consistent with the measured output and the bounded noise [2]. In contrast with the aforementioned probabilistic approaches that usually make assumptions about the statistical properties of the uncertainties, often difficult to validate, SM approaches consider a norm-bounded uncertainty. The bound of the state of the system dynamics has been considered from different perspectives among which two stand out: classical SM strategies considered ellipsoidal bounding [71, 38, 18], meanwhile in the last decades the study of zonotopes as a tool for bounding the state of a dynamical system has increased its popularity. Zonotopes were proposed to build a state bounding observer in [62] and they have been deeply studied due to their suitability. On the one hand, the fact that these sets can be represented in terms of vectors and matrices eases transmission of information between agents. On the other hand, basic operations with zonotopes are reduced to matrix calculations, simple enough to be carried out in distributed embedded systems with limited computation capabilities [55].

1.6 Thesis overview and contributions

In this section, a brief summary of the contributions of each chapter is presented.

Chapter 2 introduces some notation and preliminaries that are not the main focus of the thesis but are crucial to understand the rest of the developments. The chapter presents a decomposition of the state space in orthogonal subspaces that captures the locally observable modes of each agent involved in the network and the modes that are accessible with the information provided by the neighborhood.

In Chapter 3, an observer structure based on the subspace decomposition aforementioned is presented. Based on that decomposition, the observers use local information measured from the plant to correct the locally observable subspace, whereas the locally unobservable subspace is divided according to the "innovations" introduced by the neighboring agents, which are incorporated in the observers' dynamics through a consensus term. The design of the observer structure is tackled and it is demonstrated that the proposed design is always feasible under the given assumptions.

The proposed method introduced in Chapter 3 not only guarantees stability in the estimation for every system considered but also provides flexibility in adjusting the convergence rate of the estimation dynamics. Additionally, the approach has other positive features such as not requiring state augmentation or the resolution of linear matrix inequalities, which reduces the computational cost of the problem. Furthermore, the observer structure design is carried out in a distributed way. This feature is essential when working with large-scale networks or time-varying topologies. Lastly, the distributed design can be carried out with reasonably low information exchange, and once the estimators have been designed, the communication requirements are even lower, as the agents only need to communicate a certain portion of the state, reducing congestion of the network.

Chapter 4 discusses the same problem of Chapter 3, but in this case, it deals with perturbed systems. Based on the same observer structure as in Chapter 3, the design of the observer gains, namely, local Luenberger gain and consensus matrices, is tackled by minimizing a local quadratic cost function. This cost function, analogous with the classical LQR controller design, considers two terms: the former is inspired by the term $x^\top Qx$ that weights the deviation of the system state from the reference. In the estimation scheme considered, the first term weights the estimation error of the observer. The latter corresponds to $u^\top Ru$ in the classical control scheme. The purpose of this term is to weight the influence of the measured information in the estimation structure with respect to the system model of the plant.

This chapter demonstrates, by using linear programming, the optimality and stability of the observer in a distributed framework. Finally, the chapter presents a way to choose the weighting matrices of the cost function based on the experience of the control engineer. In particular, a scalar parameter must be chosen to tradeoff between the reliability of the model and the accuracy of the measurements.

In Chapter 5, the problem considered is somewhat different. The network of agents is intended to undertake the distributed estimation of the state of a plant by executing a distributed data-fusion-based algorithm. Data fusion is the process of integrating multiple data sources to produce more consistent, accurate, and useful information than that provided by any singular data source. Thus, every agent collects the necessary information to reconstruct the entire state.

Different from the previous chapters, this chapter considers that the agents communicate through a multihop network, where data transmitted may take several sampling instants to reach its final destination. In other words, communications are affected by graph-induced delays. Thus, the observer gains are designed to guarantee the stability of the distributed observer in spite of the presence of delays.

This data-fusion-based structure has several positive features, many of which are the same as in the observer structure of Chapters 3 and 4, such as the ability to design the observer in a distributed fashion or the reduction in the exchange of information in the network; however, additionally, this structure accounts for other contributions. First, unlike the conventional data-fusion approaches, the information is not required to be spread through the network in a single sample time, thereby relaxing the network requirements. Furthermore, the observer design reduces the exchange of information with respect to other data fusion algorithms due to the fact that the agents are not required to collect the information of every agent to reconstruct the entire state. Finally, in the case of duplicated information, the proposed subspace decomposition allows the observer to be selective when deciding which agents will be the source of the required data, thus allowing for the rejection of highly noisy information or information sent from an agent damaged by cyberattacks or malfunctioning.

Chapter 6 proposes a design method for the data-fusion-based structure introduced in the previous chapter. This design deals with the same distributed estimation problem than in Chapter 5 but, this time, considering Gaussian perturbations in the system model and agents' measurements. For this scenario, the design proposed minimizes the expected value of the estimation error norm.

Finally, Chapter 7 includes the conclusion of the thesis, where the main achievements are drawn together with potential weaknesses, limitations, and the potential future work.

1.7 List of publications supporting this thesis

The following articles have been issued or submitted for publication during the elaboration of this thesis:

Journal papers:

1. Álvaro Rodríguez del Nozal, Pablo Millán, Luis Orihuela, Alexandre Seuret and Luca Zaccarian. Distributed estimation based on multi-hop subspace decomposition. Automatica.
2. Álvaro Rodríguez del Nozal, Pablo Millán and Luis Orihuela. Data Fusion Based on Subspace Decomposition for Distributed State Estimation in Multi-Hop Networks. Sensors.
3. Álvaro Rodríguez del Nozal, Luis Orihuela and Pablo Millán. Distributed Estimation Design for LTI systems: A Linear Quadratic Approach. International Journal of Systems Science.

International Conference papers:

1. Pablo Millán, Álvaro Rodríguez del Nozal, Luca Zaccarian, Luis Orihuela and Alexandre Seuret. Distributed implementation and design for state estimation. IFAC World Congress 2017. Toulouse (France).
2. Álvaro Rodríguez del Nozal, Luis Orihuela and Pablo Millán. Distributed consensus-based Kalman filtering considering subspace decomposition. IFAC World Congress 2017. Toulouse (France).

3. Álvaro Rodríguez del Nozal, Luis Orihuela and Pablo Millán. A Game-Theoretic Framework for Distributed Voltage Regulation over HVDC grids. IEEE European Control Conference 2018. Limassol (Cyprus).

National Conference papers:

1. Álvaro Rodríguez del Nozal, Luis Orihuela, Pablo Millán, Carmelina Ierardi and Alejandro Tapia. Diseño LQ e implementación distribuida para la estimación de estado. XXXVIII Jornadas de Automática. Gijón (Asturias).

Chapter 2

Notation and preliminaries

2.1 Introduction

This chapter sums up important notation, definitions, assumptions and key lemmas that are not the main focus of the thesis but are crucial to understand the rest of the developments.

2.2 Notation

The following notation is used through the thesis:

System model:

- This thesis considers a discrete-time LTI system observed by a set of agents with the following state-space representation:

$$x^+ = Ax + w,$$
$$y_i = C_i x + n_i$$

  where $x \in \mathbb{R}^n$ is the state vector, $y_i \in \mathbb{R}^{m_i}$ is the output locally measured by each agent i, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $C_i \in \mathbb{R}^{m_i\times n}$ is the output matrix of agent i and $w \in \mathbb{R}^n$ and $n_i \in \mathbb{R}^{m_i}$ are state and measurement noises, respectively.
- Let $\hat{x}_i \in \mathbb{R}^n$ be the estimation of state x by agent i.

Graph Theory:

- A graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ comprising a set $V = \{1, 2, \ldots, p\}$ of *vertices* or *agents*, and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of *edges* or *links*.

- A directed graph is a graph in which edges have orientations, so that if $(j, i) \in \mathcal{E}$, then agent i obtains information from agent j.
- A directed path from node i_1 to node i_k is a sequence of edges such as $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ in a directed graph.
- The *neighborhood* of $i, \mathcal{N}_i \triangleq \{j : (j, i) \in \mathcal{E}\}$ is defined as the set of nodes with edges incoming to node i.
- Given $\rho \in \mathbb{Z} > 0$, the *ρ-hop reachable set* of $i, \mathcal{N}_{i,\rho}$, is defined as the set of nodes with a direct path to i involving ρ edges. Note that the 1-hop reachable set of i corresponds to the neighborhood of i.

Linear Algebra:

- The operator col$(\cdot, \cdot)$ stacks subsequent matrices into a column vector, e.g. for $A \in \mathbb{R}^{m_1 \times n}$ and $B \in \mathbb{R}^{m_2 \times n}$, col$(A, B) = [A^\top \;\; B^\top]^\top \in \mathbb{R}^{(m_1+m_2)\times n}$.
- $Im(A)$ denotes the image of matrix A, i.e., the subspace generated by the columns of matrix A. $\sigma(A)$ denotes the set of eigenvalues of matrix A.
- A collection of subspaces $\{Im(W_1), \ldots, Im(W_n)\}$ is independent if no nonzero column of W_i is a linear combination of some columns of the rest of matrices W_j, for all $j \neq i$.
- The sum of two subspaces $Im(W_1)$ and $Im(W_2)$ is denoted by $Im(W_1) + Im(W_2) = \{w_1 + w_2 | w_1 \in Im(W_1), w_2 \in Im(W_2)\}$.
- The sum of $Im(W_1)$ and $Im(W_2)$ is direct if $Im(W_1) \cap Im(W_2) = \{0\}$ and is denoted by $Im(W_1) \oplus Im(W_2)$.
- Let $||x||_\infty = \max\{|x_1|, \ldots, |x_p|\}$ be the infinity norm of vector $x = [x_1, \ldots, x_p]$.
- Let I_n denotes the identity matrix of dimension n.

2.3 Multi-hop subspace decomposition

Consider a set of agents $\mathcal{V} = \{1, 2, ..., p\}$ connected according to a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and intended to distributedly estimate the state of the following discrete-time LTI system:

$$x^+ = Ax, \tag{2.1}$$

$$y_i = C_i x \quad \forall i \in \mathcal{V}, \tag{2.2}$$

where x is the state vector, A is the system matrix, $y_i \in \mathbb{R}^{m_i}$ is the output locally measured by each agent i and $C_i \in \mathbb{R}^{m_i \times n}$ is the output matrix of agent i.

The observation structures proposed in the next chapters rely on system transformations to the observability staircase form (see for instance Theorem 16.2 in [26]). Prior to that, the following definitions are needed.

Definition 2.3.1 *The ρ-hop output matrix of agent i, $C_{i,\rho}$, is a matrix that stacks the $(\rho-1)$-hop output matrix of agent i and the $(\rho-1)$-hop output matrices of its neighborhood, $\mathcal{N}_i$. That is:*

$$C_{i,\rho} := \begin{bmatrix} C_{i,\rho-1} \\ \mathrm{col}(C_{j,\rho-1})_{j\in\mathcal{N}_i} \end{bmatrix}, \quad \forall \rho \geq 1,$$

where $C_{i,0} := C_i$.

Intuitively speaking, the ρ-hop output matrix of agent i, $C_{i,\rho}$, is composed by its output matrix C_i and the output matrices of all the agents j with a direct path to i involving ρ or less edges.

For system (2.1)-(2.2), it is well known that it is possible to find a coordinate transformation matrix, $\begin{bmatrix}\bar{V}_{i,\rho} & V_{i,\rho}\end{bmatrix} \in \mathbb{R}^{n\times n}$, according to pair $(C_{i,\rho}, A)$ such that the change of variable $\xi_{i,\rho} \triangleq \begin{bmatrix}\bar{V}_{i,\rho} & V_{i,\rho}\end{bmatrix}^\top x \in \mathbb{R}^n$ transforms the original state-space representation into the observability staircase form:

$$\xi_{i,\rho}^+ = \begin{bmatrix} \bar{V}_{i,\rho}^\top \\ V_{i,\rho}^\top \end{bmatrix} A \begin{bmatrix}\bar{V}_{i,\rho} & V_{i,\rho}\end{bmatrix} \xi_{i,\rho}.$$

Note that $\bar{V}_{i,\rho} \in \mathbb{R}^{n\times n_{i,\rho}^{\bar{o}}}$ is composed by $n_{i,\rho}^{\bar{o}}$ column vectors in $\mathbb{R}^n$ that form an orthogonal basis of the unobservable subspace of pair $(C_{i,\rho}, A)$. Correspondingly, $V_{i,\rho} \in \mathbb{R}^{n\times n_{i,\rho}^{o}}$ is an orthogonal basis of its orthogonal complement.

Definition 2.3.2 *The ρ-hop unobservable subspace from agent i, denoted $\bar{\mathcal{O}}_{i,\rho}$, is composed of all system modes that cannot be observed from the output locally measured by agent i and those measured by all the agents belonging to the s-hop reachable set of i, $\forall s \in \{0,\dots,\rho\}$. Equivalently, the ρ-hop unobservable subspace from agent i is the unobservable subspace related to pair $(C_{i,\rho}, A)$ using the above coordinate transformation:*

$$\bar{\mathcal{O}}_{i,\rho} := Im(\bar{V}_{i,\rho}).$$

The orthogonal complement of $\bar{\mathcal{O}}_{i,\rho}$, with some abuse of notation, is denoted ρ-hop observable subspace from agent i, $\mathcal{O}_{i,\rho} := Im(V_{i,\rho})$. We denote $n_{i,\rho}^o = dim(\mathcal{O}_{i,\rho})$.

According to Definition 2.3.2, it is clear that:

$$\mathcal{O}_{i,\rho-1} \subseteq \mathcal{O}_{i,\rho}, \quad \forall i \in \mathcal{V}, \quad \rho \geq 0. \tag{2.3}$$

where we consider $\mathcal{O}_{i,-1} = \emptyset$. Then, the vectors of the "innovation" basis that generates $\mathcal{O}_{i,\rho} \cap (\mathcal{O}_{i,\rho-1})^{\perp}$ can be stacked into a matrix $W_{i,\rho} \in \mathbb{R}^{n \times n_{i,\rho}}$, where $n_{i,\rho} = n^o_{i,\rho} - n^o_{i,\rho-1}$, in such a way that:

$$Im(W_{i,\rho}) := \mathcal{O}_{i,\rho} \cap (\mathcal{O}_{i,\rho-1})^{\perp}, \quad \rho \geq 0, \tag{2.4}$$

Let us, to be selected later, define $\ell_i \in \mathbb{Z}_{>0}$ as an arbitrary number of hops. From these definitions it is clear that for all $\rho \in \{0, \ldots, \ell_i\}$ and all $i \in \mathcal{V}$, it holds that

$$Im(V_{i,\rho}) = Im\left(\begin{bmatrix} W_{i,\rho} & V_{i,\rho-1} \end{bmatrix}\right), \tag{2.5}$$
$$Im(\bar{V}_{i,\rho-1}) = Im\left(\begin{bmatrix} W_{i,\rho} & \bar{V}_{i,\rho} \end{bmatrix}\right), \tag{2.6}$$

with $\bar{V}_{i,-1} := I_n$.

It is worth pointing out that $Im(W_{i,\rho})$ corresponds to the innovation introduced by the ρ-hop reachable set $\mathcal{N}_{i,\rho}$ of agent i, that is, the observable modes for agent i at hop ρ that are not observable at hop $\rho - 1$. Accordingly, the transformation matrix T_i, defined as $T_i = [\bar{V}_{i,\ell_i} \ V_{i,\ell_i}]$, can be divided using the innovations at each hop:

$$T_i := \big[\underbrace{\bar{V}_{i,\ell_i} \quad W_{i,\ell_i} \quad \cdots \quad W_{i,\rho+1}}_{\bar{V}_{i,\rho}} \quad \underbrace{W_{i,\rho} \quad \cdots \quad W_{i,0}}_{V_{i,\rho}}\big] \in \mathbb{R}^{n \times n}, \tag{2.7}$$

for all $\rho \in \{0, \ldots, \ell_i\}$, where it is easy to identify the observable and unobservable subspaces of the system by agent i at hop ρ.

It is worth pointing out that the subspace decomposition presented in this section can be altered willfully in the case that one agent does not want to consider some measurements (see for instance the example in Figure 2.1).

The following lemma introduces some important properties that are central for the subsequent derivations.

Lemma 2.3.1 *For any agent $i \in \mathcal{V}$, the next properties hold, $\forall \rho, \rho' \in \{1, \ldots, \ell_i\}$ such that $\rho \neq \rho'$:*

(i) $W_{i,\rho}^{\top} W_{i,\rho'} = 0$,
(ii) $Im(W_{j,\rho-1}) \subseteq Im(V_{i,\rho}), \ \forall j \in \mathcal{N}_i$,
(iii) $Im(W_{i,\rho}) \subseteq \bigoplus_{j \in \mathcal{N}_i} Im(W_{j,\rho-1})$,

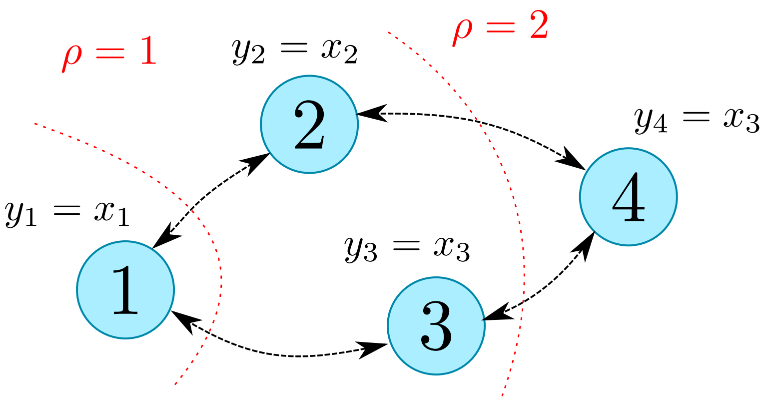

Figure 2.1: Assume that agent 1 does not want to consider y_3. Then, it can design $W_{1,1}$ to only include the basis vector of y_2 and $W_{1,2}$ to include the corresponding basis of y_3.

Proof of (i): Take any $\rho \neq \rho'$ and assume without loss of generality $\rho > \rho'$. From (2.4) we have $Im(W_{i,\rho}) \subseteq (\mathcal{O}_{i,\rho-1})^{\perp}$ and $Im(W_{i,\rho-1}) \subseteq \mathcal{O}_{i,\rho-1}$ and then, $W_{i,\rho}^{\top} W_{i,\rho-1} = 0$. Applying (2.3) recursively we obtain that $Im(W_{i,\rho})$ is orthogonal to $Im(W_{i,\rho'})$ for all $\rho' > \rho$, which proves item (i).

Proof of (ii): From Definition 2.3.2, we have that pairs $(C_{i,\rho}, A)$ and $(C_{j,\rho-1}, A)$ generate subspaces $\mathcal{O}_{i,\rho}$ and $\mathcal{O}_{j,\rho-1}$ respectively. Then, Definition 2.3.1 ensures that matrix $C_{j,\rho-1}$ is one of the stacked matrices in $C_{i,\rho}$, which clearly implies $\mathcal{O}_{j,\rho-1} \subseteq \mathcal{O}_{i,\rho}$. Finally, from (2.4) we have $Im(W_{j,\rho-1}) \subseteq \mathcal{O}_{j,\rho-1}$ and, consequently, $Im(W_{j,\rho-1}) \subseteq \mathcal{O}_{i,\rho}$ establishing item (ii).

Proof of (iii): We follow the same reasoning as for (ii). From Definition 2.3.1, we know that matrix $C_{i,\rho}$ is composed by matrix $C_{i,\rho-1}$ and matrices $C_{j,\rho-1}$ for every neighbor j of agent i. From Definition 2.3.2, it is easy to check that $\mathcal{O}_{i,\rho} = Im(V_{i,\rho-1}) \bigoplus_{j \in \mathcal{N}_i} (Im(V_{j,\rho-1}))$. Moreover, (2.4) implies that $Im(W_{i,\rho}) \subseteq (\mathcal{O}_{i,\rho-1})^{\perp}$ and $Im(W_{i,\rho}) \subseteq \mathcal{O}_{i,\rho}$, and, consequently, $Im(W_{i,\rho}) \subseteq \bigoplus_{j \in \mathcal{N}_i} Im(V_{j,\rho-1})$, which are the subspaces generated by output matrices $C_{j,\rho-1}$. Note that matrices $C_{j,\rho-2}$ are a part of $C_{i,\rho-1}$ so that $\mathcal{O}_{j,\rho-2} \subseteq \mathcal{O}_{i,\rho-1}$. This implies, using $Im(W_{i,\rho}) \subseteq (\mathcal{O}_{i,\rho-1})^{\perp}$ again, that $Im(W_{i,\rho}) \subseteq (\mathcal{O}_{j,\rho-2})^{\perp}$ for every neighbor j of i. Using (2.6), $(\mathcal{O}_{j,\rho-2})^{\perp} = Im([W_{j,\rho-1} \quad \bar{V}_{j,\rho-1}])$ and then $Im(W_{i,\rho}) \subseteq \bigoplus_{j \in \mathcal{N}_i} Im([W_{j,\rho-1} \quad \bar{V}_{j,\rho-1}])$. Since $Im(V_{j,\rho-1}) \cap Im(\bar{V}_{j\rho-1}) = \emptyset$ by definition, then item (iii) is proven. $\square$

Additionally, the sequel condition is established:

Lemma 2.3.2 *For any $i \in \mathcal{V}$ and any $j \in \mathcal{N}_{i,\rho}$, it holds $C_j W_{i,r} = 0$, with $\rho, r \in \{0, \ldots, \ell_i\}$ and $r > \rho$.*

Proof 2.3.1 *From Definition 2.3.1 it is easy to see that $Im(C_j^\top) \subseteq Im(C_{i,\rho}^\top)$ for all $j \in \mathcal{N}_{i,\rho}$. By using Definition 2.3.2, we have that pair $(C_{i,\rho}, A)$ generates the subspace $\mathcal{O}_{i,\rho}$ which directly implies that $Im(C_{i,\rho}^\top)$ belongs to $Im(V_{i,\rho})$ and consequently $Im(C_j^\top) \subseteq Im(V_{i,\rho})$. Finally, considering the orthogonality between $V_{i,\rho}$ and $W_{i,r}$ for every $r > \rho$ the Lemma is proved.* $\square$

Next, we include a definition and a necessary assumption for the solvability of the distributed estimation problem.

Definition 2.3.3 *Given $\alpha \in (0,1)$, pair (C, A) is α-detectable if pair $(C, A/\alpha)$ is detectable (in the sense of [26, Def 16.1]). Moreover, system* (2.1)-(2.2) *is collectively α-detectable if for each agent $i \in \mathcal{V}$, there exists a finite number of hops $\ell_i \in \mathbb{Z} > 0$ such that pair (C_{i,ℓ_i}, A) is α-detectable.*

In other words, pair (C, A) is α-detectable if the unobservable modes have exponential convergence at least equal to α.

By definition, we see that a pair (C, A) is α-detectable if and only if the unobservable modes of the observable decomposition have convergence rate of at least α, namely if and only if there exists an observer ensuring the exponential stabilization of the estimation error with convergence rate α. Similarly, system (2.1)-(2.2) is collectively detectable if for each agent, the complete information provided by the network (that is, the ρ-hop output matrix with ρ arbitrarily large) is sufficient to build such an observation law. Due to this fact, collective α-detectability is a necessary assumption to solve the distributed estimation problem introduced in Section 1.4 and whose resolution is tackled in the next chapters.

Assumption 2.3.3 *Given $\alpha \in (0,1)$, we assume that system* (2.1)-(2.2) *is collectively α-detectable.* $\square$

When the converge rate of the estimates is not a required objective to take in mind during the observer design procedure, we can refer to collective detectability. System (2.1)-(2.2) is collectively detectable if for each agent $i \in \mathcal{V}$, there exists a finite number of hops $\ell_i \in \mathbb{Z} > 0$ such that pair (C_{i,ℓ_i}, A) is detectable, i.e. the system is collectively α-detectable with any $\alpha < 1$. This fact implies that the unobservable modes decrease asymptotically with time with any convergence rate. For this scenario, Assumption 2.3.3 can be reformulated as follows:

Assumption 2.3.4 *We assume that system* (2.1)-(2.2) *is collectively detectable.* □

Remark 2.3.1 *If the communication graph is connected and pair* $([C_1^\top, \ldots, C_p^\top]^\top, A)$ *is* α*-detectable (as, for instance, in [25] and [80]), then Assumption 2.3.3 holds true. However, Assumption 2.3.3 is in general less restrictive, as it does not enforce connectivity of the network (see for instance the example in Figure 2.2).*

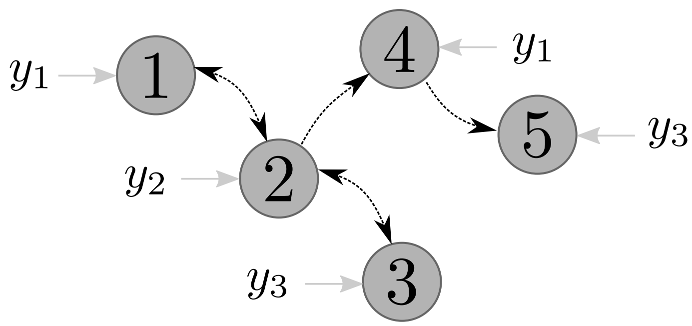

Figure 2.2: Assume that pair $(\tilde{C}, A)$ with $\tilde{C} = [C_1^\top, C_2^\top, C_3^\top]^\top$ is α-detectable. Although strong connectivity does not hold, Assumption 2.3.3 is still met.

2.4 System transformation

This section applies the linear transformation introduced in the previous section to the system considered. Prior to that, the following result that is based on Lemma 5.49 in [58] (whose straightforward proof is omitted) is presented:

Lemma 2.4.1 *If* $Im(\bar{V}_{i,\rho}) \subseteq \bar{\mathcal{O}}_{i,\rho}$*, then* $Im(A\bar{V}_{i,\rho}) \subseteq \bar{\mathcal{O}}_{i,\rho}$*, i.e., the unobservable subspace* $\bar{\mathcal{O}}_{i,\rho}$ *is an A-invariant subspace.*

Next, considering Lemma 2.4.1 and the concepts introduced in Section 2.3, the transformation of the system matrix can be obtained:

Proposition 2.4.1 *For each agent* i*, the orthogonal similarity transformation given by* T_i *in* (2.7) *transforms the system matrix* A *into a block upper-triangular matrix in the* *form:*

$$T_i^\top A T_i = \begin{bmatrix} \bar{V}_{i,\ell_i}^\top A \bar{V}_{i,\ell_i} & \bar{V}_{i,\ell_i}^\top A W_{i,\ell_i} & \cdots & \bar{V}_{i,\ell_i}^\top A W_{i,1} & \bar{V}_{i,\ell_i}^\top A W_{i,0} \\ 0 & W_{i,\ell_i}^\top A W_{i,\ell_i} & \cdots & W_{i,\ell_i}^\top A W_{i,1} & W_{i,\ell_i}^\top A W_{i,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & W_{i,1}^\top A W_{i,1} & W_{i,1}^\top A W_{i,0} \\ 0 & 0 & \cdots & 0 & W_{i,0}^\top A W_{i,0} \end{bmatrix}. \tag{2.8}$$

Proof 2.4.1 *The transformation matrix defined in* (2.7) *is composed of the vectors that form a basis of the innovations $W_{i,\rho}$, introduced by the neighbors of agent i at each hop ρ, and of those that form the basis of the unobservable subspace at hop ℓ_i, which, according to Assumption 2.3.3, must be α-detectable by the network. Note that from Lemma 2.3.1 (i), all the innovation terms are mutually orthogonal and therefore T_i is a full rank transformation matrix.*

Applying transformation T_i to the dynamics matrix of the system focusing on the partition related to hop ρ in (2.7)*, the next expression is obtained for* (2.8)*:*

$$T_i^\top A T_i = \begin{bmatrix} \bar{V}_{i,\rho}^\top A \bar{V}_{i,\rho} & \bar{V}_{i,\rho}^\top A V_{i,\rho} \\ V_{i,\rho}^\top A \bar{V}_{i,\rho} & V_{i,\rho}^\top A V_{i,\rho} \end{bmatrix}.$$

Then, according to Lemma 2.4.1, $Im(A\bar{V}_{i,\rho}) \subseteq Im(\bar{V}_{i,\rho})$ which clearly implies ${V_{i,\rho}}^\top A\bar{V}_{i,\rho} = 0$, and therefore:

$$T_i^\top A T_i = \left[\begin{array}{c|c} \bar{V}_{i,\rho}^\top A \bar{V}_{i,\rho} & \bar{V}_{i,\rho}^\top A V_{i,\rho} \\ \hline 0 & V_{i,\rho}^\top A V_{i,\rho} \end{array}\right].$$

which is valid for every considered hop ρ. Applying this procedure recursively from $\rho = 0$ to $\rho = \ell_i$, it is clear that the diagonal elements correspond to $W_{i,\rho}^\top A W_{i,\rho}$ whereas each term below the diagonal is zero, which establishes (2.8). □

Note that the first block row of matrix (2.8) corresponds to those modes that are unobservable but α-detectable.

Let us define the estimation error of any agent i as:

$$e_i := x - \hat{x}_i, \quad i \in \mathcal{V}, \tag{2.9}$$

where $\hat{x}_i \in \mathbb{R}^n$ is the estimation of the system state x made by agent i. Similarly, it is possible to define the transformed estimation error as $\varepsilon_i := T_i^\top e_i$, which can be decomposed in the transformed estimation error of agent i at each hop ρ:

$$\varepsilon_i = \begin{bmatrix} \varepsilon_{i,\ell_i+1} \\ \varepsilon_{i,\ell_i} \\ \vdots \\ \varepsilon_{i,1} \\ \varepsilon_{i,0} \end{bmatrix} = \begin{bmatrix} \bar{V}_{i,\ell_i}^\top \\ W_{i,\ell_i}^\top \\ \vdots \\ W_{i,1}^\top \\ W_{i,0}^\top \end{bmatrix} e_i. \tag{2.10}$$

Thus, due to the fact that T_i is an orthogonal matrix (and therefore $T_i^\top T_i = I_n$), the expression of the estimator error in $\varepsilon_{i,\rho}$ coordinates yields:

$$e_i = \bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1} + \sum_{r=0}^{\ell_i} W_{i,r}\varepsilon_{i,r} \tag{2.11}$$

Finally, the sequel lemma introduces an important relation that will be useful later on.

Lemma 2.4.2 *Under Assumption 2.3.4, the next equation holds for any $i \in \mathcal{V}$, any $j \in \mathcal{N}_i$, and any $\rho \in \{0, \dots, \ell_i\}$*

$$W_{j,\rho-1}^\top(\hat{x}_j - \hat{x}_i) = W_{j,\rho-1}^\top \left(\sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r} - W_{j,\rho-1}\varepsilon_{j,\rho-1} \right). \tag{2.12}$$

Proof 2.4.1 *First, let us rewrite expression* (2.12) *in terms of the estimation error defined in* (2.9)*:*

$$\begin{aligned} W_{j,\rho-1}^\top(\hat{x}_j - \hat{x}_i) &= W_{j,\rho-1}^\top(\hat{x}_j - x + x - \hat{x}_i) \\ &= W_{j,\rho-1}^\top(e_i - e_j). \end{aligned}$$

Now, consider the transformed estimation error defined in (2.10) *and let us write the estimation error of agents i and j in the $\varepsilon_{i,\rho}$, $\varepsilon_{j,\rho}$ coordinates, which yields that:*

$$W_{j,\rho-1}^\top(e_i - e_j) = W_{j,\rho-1}^\top \left(\sum_{r=0}^{\ell_i+1} W_{i,r}\varepsilon_{i,r} - \sum_{r=0}^{\ell_j+1} W_{j,r}\varepsilon_{j,r} \right).$$

According to Lemma 2.3.1 (i), we have $W_{j,\rho-1}^\top W_{j,r} = 0,\ \forall r \geq 0,\ r \neq \rho - 1$, and from Lemma 2.3.1 (ii) we know that $Im(W_{j,\rho-1}) \subseteq Im(V_{i,\rho})$ which clearly implies $W_{j,\rho-1}^\top W_{i,r} = 0,\ \forall r > \rho$, establishing the result. □

Chapter 3

Distributed estimation based on multi-hop subspace decomposition

3.1 Introduction

This chapter proposes a novel approach to the distributed estimation problem based on the subspace decomposition presented in Chapter 2. This method decomposes the state-space in orthogonal subspaces that capture the locally unobservable modes of each agent involved in the network. This work extends our preliminary result in [45], where the number of agents was limited to two. The observers use local information measured from the plant to correct the locally observable subspace, whereas the locally unobservable subspace is divided according to the innovations introduced by neighboring agents, which are incorporated in the observers' dynamics through a consensus term. The measurements and network connectivity requirements are encapsulated in a single assumption, which makes it possible to relax the common assumptions of strongly connected graphs, included for instance in [25], or strongly connected graph components, in [48] or [80].

Unlike the state augmentation approach in [59], [61] and [80], the proposed method does not require state augmentation or the resolution of linear matrix inequalities, which reduces the computational costs. More importantly, and differently from [36], [46] and [47], the design of the observers are carried out in a distributed way, which is crucial for large scale networks or time-varying topologies. Differently from [36] or [48], the presented design method provides flexibility to adjust the convergence rate of the estimation dynamics. Other positive features are that the distributed design can be carried out with a reasonably low information exchange, and that, once the estimators have been designed, the communication requirements are even lower, as the agents only need to communicate certain subspaces. Finally, the chapter demonstrates that the proposed design is always feasible under necessary assumptions.

3.2 Problem formulation

Consider a set of agents $\mathcal{V} = \{1, 2, ..., p\}$ connected according to a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and intended to distributedly estimate the state of the following discrete-time LTI system:

$$x^+ = Ax, \tag{3.1}$$

$$y_i = C_i x \quad \forall i \in \mathcal{V}. \tag{3.2}$$

In this chapter we consider an scenario free of noises and perturbations. Consider also Assumption 2.3.3, i.e. we assume that system (3.1)-(3.2) is collectively α-detectable.

3.3 Observer structure and design goal

This section presents a novel observer structure that makes use of the notions previously introduced. In particular, the proposed observer structure is as follows:

$$\widehat{x}_i^+ = \underbrace{A\widehat{x}_i}_{(a)} + \underbrace{W_{i,0}L_i(y_i - \widehat{y}_i)}_{(b)} + \underbrace{\sum_{\rho=1}^{\ell_i} \sum_{j \in \mathcal{N}_i} W_{i,\rho} N_{i,j,\rho} W_{j,\rho-1}^\top (\widehat{x}_j - \widehat{x}_i)}_{(c)} \tag{3.3}$$

where $\hat{x}_i$ is the estimation of plant state x by agent i and L_i and $N_{i,j,\rho}$ are, respectively, a local observer gain and consensus gains to be designed. The observation structure proposed in (3.3) decomposes the observer dynamics in three different terms:

(a) The first one, $A\hat{x}_i$, is the classical model-based open-loop prediction.

(b) The second term, containing $L_i(y_i - \widehat{y}_i)$, is a local Luenberger-like output injection term, intended to correct the previous prediction with the difference between the local measures and its predicted outputs $\widehat{y}_i := C_i\hat{x}_i$. It is worth noting that this term is pre-multiplied by $W_{i,0}$, which implies that the elements in the correction vector $L_i(y_i - \hat{y}_i)$ are actually used as weights to perform linear combinations of the column vectors forming $W_{i,0}$. Thus, these corrections only affect the observable subspace of agent i. This makes full sense, as the locally available output y_i only contains information about this subspace.

(c) This last term aims at adjusting the estimates $\hat{x}_i$ with the information received by the neighboring agents. Thus, the differences between the estimates of i and j are multiplied by matrix $W_{j,\rho-1}^\top$. The result is multiplied by gain matrix $N_{i,j,\rho}$ and is used as weights to perform linear combinations of $W_{i,\rho}$. It is worth mentioning that each neighbor j can compute and exchange $W_{j,\rho-1}\hat{x}_j$, whose dimension is smaller than $\hat{x}_j$, reducing in this way the exchange of information through the network.

The goal of this chapter is to design the gain matrices L_i and $N_{i,j,\rho}$ to solve the following problem:

Problem 3.3.1 *(Distributed α-estimation) Given $\alpha \in (0,1)$, plant* (3.1)-(3.2)*, and the interconnection graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, design the gains L_i and $N_{i,j,\rho}$ in* (3.3) *such that all estimates $\hat{x}_i$ converge to x exponentially fast with exponential rate α.*

3.4 Design and stability of the distributed observer

This section presents a design method for the distributed observers that guarantees stability with prescribed convergence rate.

The following property introduces the method to design the distributed observer gains. After that, it will be shown that this design guarantees exponential convergence of the estimation errors, as well as it is feasible as long as Assumption 2.3.3 is satisfied.

Property 3.4.1 *(Design of the distributed observer) For every agent i, the local observation gain L_i and the consensus gains $N_{i,j,\rho}$ are designed in such a way that for all $\rho \in \{1, \ldots, \ell_i\}$ matrices:*

$$D_{i,(0,0)} = W_{i,0}^\top A W_{i,0} - L_i C_i W_{i,0}, \tag{3.4}$$

$$D_{i,(\rho,\rho)} = W_{i,\rho}^\top A W_{i,\rho} - \sum_{j \in \mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^\top W_{i,\rho}, \tag{3.5}$$

have spectral radius smaller than α.

Based on this property, we can now state the main result of this chapter.

Theorem 3.4.1 *Consider plant (3.1) observed by a set of agents that can measure the local outputs (3.2), and that implement the observer structure defined in (3.3). If the observer gains satisfy Property 3.4.1, then the estimates of all the agents tend exponentially to the actual plant state with convergence rate* α.

Proof 3.4.1 *Let us write the transformed estimation error dynamics for agent* i *at hop* 0*:*

$$\varepsilon_{i,0}^{+} = W_{i,0}^{\top} x^{+} - W_{i,0}^{\top} \hat{x}_i^{+} = \left[W_{i,0}^{\top} A W_{i,0} - L_i C_i W_{i,0} \right] \varepsilon_{i,0}.$$

Thus, since gain L_i *is designed in such a way that matrix* (3.4) *has spectral radius smaller than* α*, then the estimation error of the locally observable modes of agent* i *tends exponentially to zero with the decay rate* α*. Note that the locally observable states of each agent are completely decoupled from the unobservable ones.*

The second part of the proof consists in proving that, if (3.5) *have spectral radius smaller than* α*, then also the estimation error of the rest of the modes converges to zero exponentially fast with rate* α*. Let us write the transformed estimation error dynamics for an agent* i *at hop* ρ*, with* $\rho \geq 1$*, using the orthogonality in Lemma 2.3.1 (i):*

$$\begin{aligned} \varepsilon_{i,\rho}^{+} &= W_{i,\rho}^{\top} \left(A e_i - W_{i,\rho} \sum_{j \in \mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^{\top} (\hat{x}_j - \hat{x}_i) \right) \\ &= W_{i,\rho}^{\top} A \sum_{r=0}^{\rho} W_{i,r} \varepsilon_{i,r} - \sum_{j \in \mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^{\top} \left(\hat{x}_j - \hat{x}_i \right), \end{aligned}$$

where $W_{i,\rho}^{\top} W_{i,\rho} = I$ *and Lemma 2.4.1 has been used to obtain* $W_{i,\rho}^{\top} A \sum_{r=\rho+1}^{\ell_i+1} W_{i,r} = 0$*. Next, thanks to Lemma 2.4.2, after some manipulations it is possible to rewrite the equation above as:*

$$\varepsilon_{i,\rho}^{+} = \sum_{r=0}^{\rho} D_{i,(\rho,r)} \varepsilon_{i,r} + \sum_{j \in \mathcal{N}_i} N_{i,j,\rho} \varepsilon_{j,\rho-1}, \tag{3.6}$$

with

$$D_{i,(\rho,r)} := \left(W_{i,\rho}^{\top} A - \sum_{j \in \mathcal{N}_j} N_{i,j,\rho} W_{j,\rho-1}^{\top} \right) W_{i,r}, \tag{3.7}$$

which extends and completes the equations (3.4) *and* (3.5)*. From* (3.6)*, it can be seen that the evolution of the estimation error of agent* i *at hop* ρ *depends on the estimation error of that agent at the previous hops and the estimation error of the neighbors at hop* $\rho - 1$*, thus revealing a cascade structure.*

We are now in the position to create a vector that stacks the estimation errors of all the agents involved in the network at each hop ρ:

$$\varepsilon_\rho := \mathrm{col}(\varepsilon_{i,\rho})_{i\in\mathcal{V}:\ell_i+1\geq\rho}, \quad \forall\rho\in\{0,\dots,\bar{\ell}\}, \tag{3.8}$$

where $\bar{\ell} = 1 + \max_{i\in\mathcal{V}} \ell_i$.

Combining (3.6) *and* (3.8)*, we generate an expression of the estimation error evolution of all the agents of the network at every hop ρ. This leads to:*

$$\begin{bmatrix}\varepsilon_{\bar{\ell}}\\ \vdots\\ \varepsilon_1\\ \varepsilon_0\end{bmatrix}^+ = \begin{bmatrix}\triangle_{\bar{\ell}} & \cdots & \star & \star\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & \triangle_1 & \star\\ 0 & \cdots & 0 & \triangle_0\end{bmatrix}\begin{bmatrix}\varepsilon_{\bar{\ell}}\\ \vdots\\ \varepsilon_1\\ \varepsilon_0\end{bmatrix}, \tag{3.9}$$

where $\star$ represents some possibly nonzero terms given by (3.6)*. It is worth pointing out that according to* (3.6) *the diagonal terms of* (3.9) *for $\rho \geq 0$ are block diagonal terms with the next structure:*

$$\triangle_\rho = \begin{bmatrix}D_{1,(\rho,\rho)} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & D_{p,(\rho,\rho)}\end{bmatrix}, \tag{3.10}$$

where $\{1,\dots,p\} \in \mathcal{V}$*, and $D_{i,(\rho,\rho)}$ for $\rho = \{1,\dots,\ell_i\}$ are defined in* (3.4)-(3.5) *(see also* (3.7)*), that is, $D_{i,(\ell_i+1,\ell_i+1)} = \bar{V}_{i,\ell_i}^\top A\bar{V}_{i,\ell_i}$ and $D_{i,(\rho,\rho)}$ is the empty matrix for all $\rho \in \{\ell_i+2,\dots,\bar{\ell}\}$.*

Thus, the eigenvalues of the upper triangular matrix in (3.9) *are the eigenvalues of the corresponding matrices placed in its diagonal, which are defined in* (3.10)*. Finally, it is clear that, if matrices* (3.4) *and* (3.5) *have spectral radius smaller than α for every agent and hop considered, then the matrix exposed in* (3.9) *also has a spectral radius smaller than α, and consequently the estimation error of every agent tends exponentially to zero with speed of convergence α.* □

Remark 3.4.1 *The distributed observer design presented has the advantage of inducing linear error dynamics for which we may obtain quadratic Lyapunov certificates. Then we may consider robust extensions of the nominal exponential stability established in Theorem 3.4.1 by relying on the intrinsic robustness of Lyapunov-based results. Among other things, this may comprise taking into account sufficiently rare packet losses and their Lyapunov characterization as in [27], which would clearly not destroy the established exponential convergence. Additionally, we may consider partially desynchronized nodes* *or sufficiently small delays in a sampled-data context where plant* (3.1) *is the sampled version of a continuous-time dynamics. Such extensions are left as future work.*

Other non-immediate extensions concern the resilience of the observer to time-varying graphs [81] or switching networks [78], and cyber-attacks on sensors [49].

It is important to note that, according to the transformation made to the system, the proposed structure for the estimators decomposes the influence of the observation gains, L_i, which only affects the locally observable subspace, from the influence of the consensus gains, $N_{i,j,\rho}$, which has an effect on the locally unobservable subspace.

It is well-known that there always exists a gain L_i able to stabilize pair $\left(C_i W_{i,0}, W_{i,0}^\top A W_{i,0}\right)$ with arbitrary spectral radius. However, it is necessary to prove the existence of matrices $N_{i,j,\rho}$ that induce the same properties on matrices (3.5). This is proved below.

Theorem 3.4.2 *(Design feasibility) It is always possible, under Assumption 2.3.3, to find a set of matrices L_i and $N_{i,j,\rho}$ that satisfy Property 3.4.1.*

Proof 3.4.2 *The existence of L_i is a well-known consequence of observability of pair $(C_i W_{i,0}, W_{i,0}^\top A W_{i,0})$ (see, e.g., the dual statement in [26, Th 12.7]). In what regards (3.5), let us fix an arbitrary i and $\rho \leq \ell_i$ and rewrite the matrix in (3.5) in the following compact form:*

$$W_{i,\rho}^\top A W_{i,\rho} - \bar{N}_{i,\rho} \Lambda_{i,\rho}, \tag{3.11}$$

where

$$\begin{aligned} \bar{N}_{i,\rho} &= \mathrm{col}(N_{i,j,\rho}^\top)_{j \in \mathcal{N}_i}^\top, \\ \Lambda_{i,\rho} &= \mathrm{col}(W_{j,\rho-1}^\top)_{j \in \mathcal{N}_i} W_{i,\rho}. \end{aligned}$$

To complete the proof it is enough to show that pair $\left(\Lambda_{i,\rho}, W_{i,\rho}^\top A W_{i,\rho}\right)$ is observable and apply the same reasoning used for (3.4).

From the Popov-Belevitch-Hautus test (see, e.g., [26, Th 15.9]) pair $(\Lambda_{i,\rho}, A_{i,\rho}^o) := (\Lambda_{i,\rho}, W_{i,\rho}^\top A W_{i,\rho})$ is observable if and only if:

$$rank \begin{bmatrix} A_{i,\rho}^o - \lambda I \\ \Lambda_{i,\rho} \end{bmatrix} = n_{i,\rho}, \quad \forall \lambda \in \sigma(A_{i,\rho}^o). \tag{3.12}$$

This condition can be guaranteed if:

$$rank\,[\Lambda_{i,\rho}] = n_{i,\rho}, \tag{3.13}$$

which clearly implies (3.12). *From Lemma 2.3.1, we know that:*

$$Im(W_{i,\rho}) \subseteq \bigoplus_{j\in\mathcal{N}_i} Im(W_{j,\rho-1}),$$

which implies that $rank(W_{i,\rho}) \leq rank(\mathrm{col}(W_{j,\rho-1}^\top)_{j\in\mathcal{N}_i})$. *Finally, using the fact that* $W_{i,\rho}$ *is a full rank matrix with* $rank(W_{i,\rho}) = n_{i,\rho}$ *it is a simple matter to check that* $rank(\mathrm{col}(W_{j,\rho-1}^\top)_{j\in\mathcal{N}_i} W_{i,\rho}) = rank(\Lambda_{i,\rho}) = rank(W_{i,\rho}) = n_{i,p}$, *which establishes* (3.13) *and completes the proof.* □

3.5 Continuous-time construction

The design technique proposed in this chapter can be easily extended to the continuous-time case. This section summarizes its adaptation to this case.

The state-space representation for the continuous-time linear time invariant system is defined as follows:

$$\begin{aligned}\dot{x} &= Ax,\\ y_i &= C_i x \quad \forall i \in \mathcal{V}.\end{aligned}$$

The subspaces definition as well as Assumption 2.3.3 in Section 2.3 are valid also for the continuous-time case. Regarding the observer structure described in Section 3.3, the continuous-time expression is given by:

$$\begin{aligned}\dot{\widehat{x}}_i &= A\widehat{x}_i + W_{i,0}L_i(y_i - \widehat{y}_i) + \\ &+ \sum_{\rho=1}^{\ell_i}\sum_{j\in\mathcal{N}_i} W_{i,\rho}N_{i,j,\rho}{W_{j,\rho-1}}^\top(\widehat{x}_j - \widehat{x}_i).\end{aligned}$$

The lemmas and propositions introduced in Section 3.4 also apply to this case. Nevertheless, Property 3.4.1 must be redefined to design local and consensus gains ensuring convergence abscissa equal to α for the matrices (3.4) and (3.5) with $\alpha \in (-\infty, 0)$. Finally, Theorems 3.4.1 and 3.4.2 apply in both cases.

3.6 Distributed design and operation

This section presents an algorithm to build the estimation structure of each agent i. This structure depends on integer and matrices $W_{i,\rho}$, $\rho = 1, \ldots, \ell_i$, which should be determined before designing the observer gains $(L_i, N_{i,j,\rho})$ according to the previous section. Finally, in the running phase, the agents estimate online (distributedly) the plant state x. These three phases are clarified below.

Distributed observer setup. In this phase, we design ℓ_i and matrices $W_{i,\rho}$. First, each agent identifies its locally observable subspace through its output matrix and the plant dynamics, and constructs matrices $V_{i,0} = W_{i,0}$. Secondly, each agent exchange its $W_{i,0}$ matrix with its neighbors. These matrices are used to define the 1-hop observable subspace and to construct matrices $W_{i,1}$. The algorithm is repeated until hop $\rho = \ell_i$ is reached. Note that whenever ℓ_i is not known, it can be assessed locally by computing an observable decomposition and checking whether the ρ-hop unobservable modes have speed of convergence faster than α. The pseudocode of the design algorithm is given next:

- For every agent i do:

    a. Compute $\mathcal{O}_{i,0}$ and construct matrix $W_{i,0}$. Set $\rho = 0$.
    b. Perform the two steps:
        * Exchange $W_{i,\rho}$ with the neighbors.
        * Construct $\mathcal{O}_{i,\rho+1}$ and construct matrix $W_{i,\rho+1}$.
    c. If the ρ-hop unobservable modes have speed α, then stop and set $\ell_i = \rho$. Otherwise increment ρ and go to (b).
    d. Exchange ℓ_i with the neighborhood $\mathcal{N}_i$.

Summarizing, in the first phase each agent i exchanges with its neighbors matrices $W_{i,\rho}$ for $\rho = \{0, \dots, \ell_i\}$. Due to the fact that, from (2.7), the set of all these matrices, together with $\bar{V}_{i,\ell_i}$ forms the transformation matrix T_i, it is clear that the exchange of information, in terms of transmitted scalars, is at most n^2.

Gain selection phase. In this phase each agent selects gains $(L_i, N_{i,j,\rho})$ in such a way that Property 3.4.1 holds. This selection does not require any information exchange because for each agent i, matrices $W_{i,\rho}$ and matrices $W_{j,\rho}$ of all the neighbors $j \in \mathcal{N}_i$ have been selected and stored in the previous phase.

Running phase. This is the online phase where the distributed observer estimates state x. Here, according to the observer structure (3.3), each agent i exchanges with all its neighbors $j \in \mathcal{N}_i$ a portion of the state defined by $W_{i,\rho-1}^{\top}\hat{x}_i$ for every $\rho \in \{1, \dots, \ell_j\}$, whose size, according to (2.7), is at most equal to n.

3.7 Simulation results

In this section a simulation example is presented in order to show the effectiveness of the proposed observer.

Example 3.7.1 *The algorithm presented will be compared with the observer structure introduced in [36]. Consider the following continuous-time system:*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} x,$$

which is being observed by four agents in such a way that $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$ *and* $y_4 = x_4$. *A cyclic topology for the connection of the agents is considered, namely the graph is composed by edges* $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 1)$.*To carry out the observer design, the system has been discretized with a sampling time of 1s.*

Figure 3.1 shows the evolution of the plant state modes and the agent 1 *estimates for the proposed distributed observer. Note that for this agent, a local design is made to estimate* x_1 *while the rest of the states are estimated thought the information provided by its neighborhood. The design has been carried out placing the local and consensus poles around* -3.

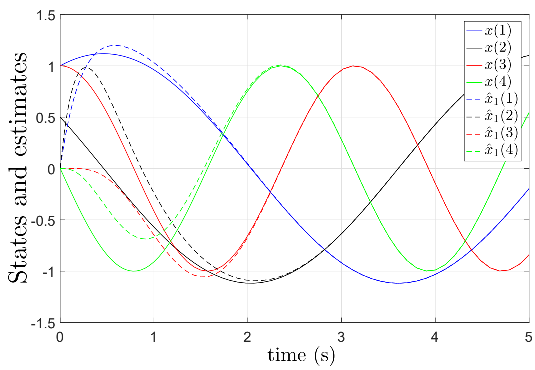

Figure 3.1: State of the plant in solid lines and estimates of agent 1 in dashed lines.

To provide a comparison between the estimation performance of the proposed observer and that corresponding to [36], the simulation parameters are set as in the simulation examples provided in that work. Figure 3.2 shows the evolution of the total error for both structures, which is defined as the average of the 2-norm of the estimation error of each agent:

$$\bar{e}(t) = \frac{1}{4} \sum_{i=1}^{4} \| x(t) - \hat{x}_i(t) \|_2 \,. \tag{3.14}$$

Different simulations have been provided regarding the convergence rate fixed by α. The observer parameters for the other algorithm has been selected from the simulation example in [36]. Finally, the initial value for the estimations of every state and agent is zero.

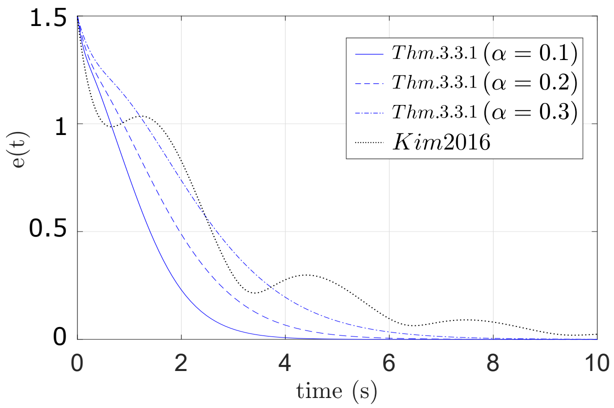

Figure 3.2: Evolution of the total error defined in (3.14) for the algorithm presented in this paper and the observer presented in [36].

It is worth pointing out that the design method proposed in Property 3.4.1 makes it easy to place conveniently the estimation error dynamics eigenvalues, taking into account the observable modes at its corresponding hops, in such a way that the convergence can be accelerated, as shown in Figure 3.2.

3.8 Conclusions

The observer structure presented in this chapter introduces a novel method to design and analyze the distributed estimation problem. By decomposing the state-space of each agent in locally observable and unobservable subspaces, the last one composed by the innovation introduced by each neighbor at each hop, a distributed design method for the observers has been developed. This design can be carried out through using simple pole placement algorithms, and allows one to adjust the convergence rate. Stability of the presented observer structure has been proven, together with a feasibility condition requiring only necessary conditions for distributed detectability.

Chapter 4

An LQ-based design method for distributed estimation

4.1 Introduction

This chapter deals with the problem of estimating the state of a plant by a network of agents executing a cooperative algorithm. Each distributed device knows the model of the plant and can access to some information of the system state. The portion of the state not accessible by itself is obtained through the exchange of information with the rest of the agents involved in the network. To do that, this chapter relays in the observer structure presented in the previous chapter. This structure allows identifying the observable modes of each agent i considering the information provided by the agents located a number of hops ρ from itself.

Thus, the chapter contributes to the problem of distributed estimation of LTI perturbed systems by proposing a novel design method for the observer structure presented in Chapter 3 (3.3). The main contributions of the chapter are enumerated next:

1. The design of the observer gains, namely local Luenberger gain and consensus matrices, is tackled by minimizing a local quadratic cost function.
2. By using linear programming, the optimality and stability of the observer is proven for a distributed framework.
3. The chapter presents a way to choose the weighting matrices of the cost function based on the experience of the control engineer. In particular, a scalar parameter must be chosen to trade off the reliability of the model and the accuracy of the measurements.

4.2 Problem formulation

Consider a set of agents $\mathcal{V} = \{1, 2, ..., p\}$ connected through a communication network characterized by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and intended to distributedly estimate the state $x \in \mathbb{R}^n$ of the following LTI system:

$$\begin{aligned} x^+ &= Ax + w, && (4.1) \\ y_i &= C_i x + n_i, \quad \forall i \in \mathcal{V}, && (4.2) \end{aligned}$$

where $w \in \mathbb{R}^n$ and $n_i \in \mathbb{R}^{m_i}$ are state and measurement noises at time k, respectively.

Assumption 4.2.1 *The norms of the noises $w(k)$ and $n_i(k)$ are upper-bounded as follows:*

$$||w(k)|| < \delta_w, \quad ||n_i(k)|| < \delta_{n_i}, \ \forall k, i \in \mathcal{V},$$

where $||\cdot||$ is a consistent norm and $\delta_w, \delta_{n_i} \in \mathbb{R}^+$ are the bounds.

Additionally, Assumption 2.3.4 is a necessary assumption, i.e. we assume that system (4.1)-(4.2) is collectively detectable.

4.3 Observer Structure

Consider the observer structure presented in Chapter 3:

$$\widehat{x}_i^+ = A\widehat{x}_i + W_{i,0}L_i(y_i - \widehat{y}_i) + \sum_{\rho=1}^{\ell_i} \sum_{j \in \mathcal{N}_i} W_{i,\rho} N_{i,j,\rho} W_{j,\rho-1}^\top (\widehat{x}_j - \widehat{x}_i). \quad (4.3)$$

The goal of this chapter is to design the gain matrices L_i and $N_{i,j,\rho}$ to solve the following problem:

Problem 4.3.1 *(Distributed LQ-design) Given plant* (4.1)-(4.2)*, and the interconnection graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, design the gains L_i and $N_{i,j,\rho}$ in* (4.3) *such that all estimates $\hat{x}_i$ converge to x asymptotically achieving a compromise between fast convergence rate and good noise rejection.*

Prior to that, the sequel proposition, that will be useful later on, defines the estimation error dynamics for the system and observer considered:

Proposition 4.3.1 *Consider the network of agents described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where every agent i implements the observer structure* (4.3) *to estimate the state of the system* (4.1). *Then, the transformed estimation error dynamics at every hop ρ is given by the following equations:*

$$\begin{aligned} \varepsilon_{i,0}^{+} &= (W_{i,0}^{\top} A W_{i,0} - L_i C_i W_{i,0})\varepsilon_{i,0} + W_{i,0}^{\top} w - L_i n_i, && (4.4)\\ \varepsilon_{i,\rho}^{+} &= \sum_{r=0}^{\rho} \left(W_{i,\rho}^{\top} A - \sum_{j\in\mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^{\top} \right) W_{i,r}\varepsilon_{i,r} + W_{i,\rho}^{\top} w && (4.5)\\ &+ \sum_{j\in\mathcal{N}_i} N_{i,j,\rho}\varepsilon_{j,\rho-1}, \end{aligned}$$

for all $\rho = \{1, \dots, \ell_i\}$.

Proof 4.3.1 *Let us write first the evolution of the estimation error dynamics for system* (4.1) *under the observation structure in* (4.3)*:*

$$\begin{aligned} e_i^{+} &= x^{+} - \hat{x}_i^{+} = Ae_i + w - W_{i,0} L_i C_i e_i - W_{i,0} L_i n_i \\ &- \sum_{\rho=1}^{\ell_i} \sum_{j\in\mathcal{N}_i} W_{i,\rho} N_{i,j,\rho} W_{j,\rho-1}^{\top} (\hat{x}_j - \hat{x}_i). \end{aligned}$$

Using (2.10), *we can write the transformed estimation error dynamics for agent i at hop* 0*:*

$$\varepsilon_{i,0}^{+} = W_{i,0}^{\top} e_i = W_{i,0}^{\top} A e_i - L_i C_i e_i + W_{i,0}^{\top} w - L_i n_i,$$

where $W_{i,0}^{\top} W_{i,0} = I_{n_{i,0}}$ and Lemma 2.3.1 (i) has been used. Next, thanks to equation (2.11), *we know the expression of e_i in $\varepsilon_{i,\rho}$ coordinates and using Lemma 2.3.1 (iv) it implies that $W_{i,0}^{\top} A \left(\bar{V}_{i,\ell_i} \bar{\varepsilon}_{i,\ell_i} + \sum_{r=1}^{\ell_i} W_{i,r}\varepsilon_{i,r} \right) = 0$ so we can rewrite the equation above as:*

$$\varepsilon_{i,0}^{+} = W_{i,0}^{\top} A W_{i,0}\varepsilon_{i,0} - L_i C_i W_{i,0}\varepsilon_{i,0} + W_{i,0}^{\top} w - L_i n_i,$$

which is the desired equation exposed in (4.4).

The second part of the proof consists in obtaining expression (4.5). *Let us write the transformed estimation error dynamics for agent i at hop ρ, with $\rho \geq 0$. Using the orthogonality in Lemma 2.3.1 (i):*

$$\varepsilon_{i,\rho}^{+} = W_{i,\rho}^{\top} e_i^{+} = W_{i,\rho}^{\top} A e_i + W_{i,\rho}^{\top} w - \sum_{j\in\mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^{\top} (\hat{x}_j - \hat{x}_i),$$

where we have applied $W_{i,\rho}^\top W_{i,\rho} = I_{n_{i,\rho}}$. *Next, anagolously as with* $\rho = 0$, *thanks to equation* (2.11), *we can substitute* e_i *in* $\varepsilon_{i,\rho}$ *coordinates and using Lemma 2.3.1 (iv) we know that* $W_{i,\rho}^\top A \left(\bar{V}_{i,\ell_i} \bar{\varepsilon}_{i,\ell_i} + \sum_{r=\rho+1}^{\ell_i} W_{i,r}\varepsilon_{i,r} \right) = 0$. *Hence, the above equation yields:*

$$\varepsilon_{i,\rho}^{+} = W_{i,\rho}^\top A \sum_{r=0}^{\rho} W_{i,r} + W_{i,\rho}^\top w - \sum_{j\in\mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^\top (\hat{x}_j - \hat{x}_i).$$

Finally, applying Lemma 2.4.2, equation (4.5) *is obtained, thus completing the proof.* □

4.4 LQ based observer design

This section presents an LQ-based design for the observers in (4.3). It is first shown that the proposed design guarantees optimality and asymptotic convergence in the absence of plant and measurement noises. Then, it is demonstrated that Assumption 2.3.4 suffices to ensure the feasibility of the proposed design. After that, it is shown that the dynamics of the estimation error is GUUB in the presence of noises. Finally, a tuning method is proposed to choose the weights of the cost functions associated to the LQ-design.

The evolution of a dynamical system $x^+ = Ax$ is GUUB with ultimate bound b if it exists positive constants b and c, independent of $t_0 \geq 0$, such that for every $0 < a < c$ arbitrarily large, there exist a $T = T(a,b) > 0$, independent of t_0, such that $||x(t_0)|| \leq a \Rightarrow ||x(t)|| \leq b$, for all $t \geq t_0 + T$.

4.4.1 Design of the distributed observer

Let us consider the following local quadratic cost function:

$$J_i(k) = \sum_{\rho=0}^{\ell_i} \sum_{t=k}^{\infty} \left(\varepsilon_{i,\rho}(t)^\top U_{i,\rho} \varepsilon_{i,\rho}(t) + u_{i,\rho}(t)^\top S_{i,\rho} u_{i,\rho}(t) \right), \tag{4.6}$$

where

$$u_{i,0}(t) = -L_i C_i W_{i,0} \varepsilon_{i,0}(t), \tag{4.7}$$

$$u_{i,\rho}(t) = -\sum_{j\in N_i} N_{i,j,\rho} {W_{j,\rho-1}}^\top W_{i,\rho} \varepsilon_{i,\rho}(t), \quad \rho = \{1, \ldots, \ell_i\}, \tag{4.8}$$

and $U_{i,\rho} \in \mathbb{R}^{n_{i,\rho} \times n_{i,\rho}}$ and $S_{i,\rho} \in \mathbb{R}^{n_{i,\rho} \times n_{i,\rho}}$ are diagonal positive definite weighting matrices.

The term $\varepsilon_{i,\rho}(t)^\top U_{i,\rho}\varepsilon_{i,\rho}(t)$ in (4.6) is the stabilization cost of the estimation error, computed for every hop ρ. By analogy with the classical cost functions in LQ control problems, this term is inspired by the term $x^\top Qx$ that weights the deviation of the system state/estimation error from the reference. The purpose of term $u_{i,\rho}(t)^\top S_{i,\rho}u_{i,\rho}(t)$ is to weight, on the one hand, the information feedback at hop 0, which involves only the local measured plant output and the corresponding gain L_i, and on the other hand, the information feedback at further hops, involving neighbors estimates and consensus matrices $N_{i,j,\rho}$. Using the same analogy, it weights the influence of the feedback signal as the term $u^\top Ru$, typically used in LQ control.

Property 4.4.1 *For every agent $i \in \mathcal{V}$, the estimation gains L_i and $N_{i,j,\rho}$ are designed in such a way that for all $\rho \in \{0,\dots,\ell_i\}$:*

$$\left(S_{i,0} + P_{i,0}\right)^{-1} P_{i,0}W_{i,0}^\top AW_{i,0} = L_iC_iW_{i,0}, \tag{4.9}$$

$$\left(S_{i,\rho} + P_{i,\rho}\right)^{-1} P_{i,\rho}W_{i,\rho}^\top AW_{i,\rho} = \sum_{j\in N_i} N_{i,j,\rho}W_{j,\rho-1}{}^\top W_{i,\rho}, \tag{4.10}$$

for all $\rho \in \{0,\dots,\ell_i\}$, where $P_{i,\rho} \in \mathbb{R}^{n_{i,\rho}\times n_{i,\rho}}$ are positive definite matrices solution of the following discrete-time algebraic Riccati equation:

$$\begin{aligned} &W_{i,\rho}^\top A^\top W_{i,\rho}P_{i,\rho}W_{i,\rho}^\top AW_{i,\rho} - P_{i,\rho} + U_{i,\rho} \\ &\qquad = W_{i,\rho}^\top A^\top W_{i,\rho}P_{i,\rho}\left(S_{i,\rho} + P_{i,\rho}\right)^{-1} P_{i,\rho}W_{i,\rho}^\top AW_{i,\rho}, \end{aligned} \tag{4.11}$$

for all $\rho \in \{0,\dots,\ell_i\}$.

Based on this property, we can now state the main result of the chapter.

Theorem 4.4.2 *Consider system* (4.1) *in the absence of plant and measurements noises, and the observation structure defined in* (4.3). *Then,*

1. *If all the estimation errors $\varepsilon_{i,r}$ converge to zero for $0 \le r < \rho$, $\forall i \in \mathcal{V}$, the gain matrices L_i and $N_{i,j,\rho}$ that minimize the cost function* (4.6) *at hop ρ are given by Property 4.4.1.*
2. *If the estimations gains L_i and $N_{i,j,\rho}$ for every $\rho \in \{0,\dots,\ell_i\}$ are designed satisfying Property 4.4.1, then the estimates of all the agents tend asymptotically to the actual plant state.*

Proof 4.4.1 *First, it will be shown that, provided that all the estimation errors $\varepsilon_{i,r}$ converge to zero for $0 \le r < \rho$, $\forall i \in \mathcal{V}$, the optimal design of the estimation gains are given by* (4.9)-(4.11). *After that, the asymptotic stability will be proven by induction.*

For the first part, let us write the dynamics of $\varepsilon_{i,\rho}$ *according to* (4.5) *in the absence of noises and with* $\varepsilon_{i,r} \equiv 0, \forall r : 0 \leq r < \rho$*:*

$$\varepsilon_{i,\rho}^{+} = W_{i,\rho}^{\top} A W_{i,\rho} \varepsilon_{i,\rho} - \sum_{j \in \mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^{\top} W_{i,\rho} \varepsilon_{i,\rho} = W_{i,\rho}^{\top} A W_{i,\rho} \varepsilon_{i,\rho} + u_{i,\rho}, \tag{4.12}$$

where (4.8) *has been used.*

Now, let us write the cost function in (4.6) *as* $J_i(k) = \sum_{\rho=0}^{\ell_i} J_{i,\rho}(k)$, *with* $J_{i,\rho}(k) = \sum_{t=k}^{\infty} \Big(\varepsilon_{i,\rho}(t)^{\top} U_{i,\rho} \varepsilon_{i,\rho}(t) + u_{i,\rho}(t)^{\top} S_{i,\rho} u_{i,\rho}(t) \Big)$.

Given the quadratic dependence of $\varepsilon_{i,\rho}(k)$, *it is clear that the values* $u_{i,\rho}(t)$ *that minimize* $J_{i,\rho}(k)$ *are linear functions of* $\varepsilon_{i,\rho}(k)$, *and therefore it is possible to write the optimal costs of each agent as:*

$$J_{i,\rho}{}^{*}(k) = \varepsilon_{i,\rho}(k)^{\top} P_{i,\rho} \varepsilon_{i,\rho}(k), \tag{4.13}$$

for some $P_{i,\rho} \in \mathbb{R}^{n_{i,\rho} \times n_{i,\rho}} > 0$. *Furthermore:*

$$\begin{aligned} J_{i,\rho}(k) = \kappa_{i,\rho}(\varepsilon_{i,\rho}, u_{i,\rho}, k) + \sum_{t=k+1}^{\infty} \kappa_{i,\rho}(\varepsilon_{i,\rho}, u_{i,\rho}, t) & \\ = \kappa_{i,\rho}(\varepsilon_{i,\rho}, u_{i,\rho}, k) + J_{i,\rho}(k+1), & \end{aligned} \tag{4.14}$$

where $\kappa_{i,\rho}(\varepsilon_{i,\rho}, u_{i,\rho}, k) := \varepsilon_{i,\rho}(k)^{\top} U_{i,\rho} \varepsilon_{i,\rho}(k) + u_{i,\rho}(k)^{\top} S_{i,\rho} u_{i,\rho}(k)$.

To compute the optimal $u_{i,\rho}^{*}(k)$ *for which the minimum cost* $J_{i,\rho}^{*}(k)$ *is attained, let us rewrite the equation above using* (4.13), *which results in:*

$$\begin{array}{rcl} J_{i,\rho}(k) & = & \varepsilon_{i,\rho}(k)^{\top} U_{i,\rho} \varepsilon_{i,\rho}(k) + u_{i,\rho}(k)^{\top} S_{i,\rho} u_{i,\rho}(k) \\ & + & \varepsilon_{i,\rho}(k+1)^{\top} P_{i,\rho} \varepsilon_{i,\rho}(k+1) \\ & = & \varepsilon_{i,\rho}(k)^{\top} U_{i,\rho} \varepsilon_{i,\rho}(k) + u_{i,\rho}(k)^{\top} S_{i,\rho} u_{i,\rho}(k) \\ & + & (\varepsilon_{i,\rho}(k)^{\top} W_{i,\rho}^{\top} A^{\top} W_{i,\rho} + u_{i,\rho}(k)^{\top}) P_{i,\rho} \\ & \times & (W_{i,\rho}^{\top} A W_{i,\rho} \varepsilon_{i,\rho}(k) + u_{i,\rho}(k)). \end{array} \tag{4.15}$$

Thus $u_{i,\rho}^{*}(k) = \arg \min_{u_{i,\rho}(k)} J_{i,\rho}(k) = u_{i,\rho}(k) : \dfrac{\partial J_{i,\rho}(k)}{\partial u_{i,\rho}(k)} = 0$, *which yields to:*

$$u_{i,\rho}^{*}(k) = -\left(S_{i,\rho} + P_{i,\rho}\right)^{-1} P_{i,\rho} W_{i,\rho}^{\top} A W_{i,\rho} \varepsilon_{i,\rho}(k). \tag{4.16}$$

Substituting (4.16) *in* (4.15), *it is straightforward to obtain* (4.11), *from which* $P_{i,\rho}$ *can be computed. Then, by comparing* (4.16) *and* (4.8), *it is clear that the observer gains must be designed according to equation* (4.10).

Now let us move to the second claim of the theorem. First of all, we assume that all the estimation errors $\epsilon_{i,r}$ *converge to zero for* $0 \leq r < \rho$, $\forall i \in \mathcal{V}$, *and this lead us to prove the stability of* $\varepsilon_{i,\rho}$. *Later, it will be proven the convergence of* $\epsilon_{i,0}$.

To show the stabilization of the error $\varepsilon_{i,\rho}$, *consider that from* (4.14) *it is directly obtained that* $J_{i,\rho}(k+1) - J_{i,\rho}(k) = -\kappa_{i,\rho}(\varepsilon_{i,\rho}, u_{i,\rho}, k)$. *Thus, taking as a Lyapunov function* $V_{i,\rho}(k) = J^*_{i,\rho}(k) = \varepsilon_{i,\rho}(k)^\top P_{i,\rho}\varepsilon_{i,\rho}(k)$, *it holds that* $\Delta V_{i,\rho}(k) = V_{i,\rho}(k+1) - V_{i,\rho}(k) = -\kappa_{i,\rho}(\varepsilon_{i,\rho}, u_{i,\rho}, k) < 0$, *which ensures the asymptotic convergence of* $\varepsilon_{i,\rho}$ *to the origin in absence of noise.*

Finally, it suffices to show that in the absence of noises the proposed design guarantees the stabilization of $\varepsilon_{i,\rho}$ *for* $\rho = 0$. *From* (4.4), *it yields that* $\varepsilon^+_{i,0} = (W^\top_{i,0}AW_{i,0} - L_iC_iW_{i,0})\varepsilon_{i,0}$. *Repeating the same procedure above, it is easily obtained that* $u^*_{i,0}(k) = -\left(S_{i,0} + P_{i,0}\right)^{-1} P_{i,0}W^\top_{i,0}AW_{i,0}\varepsilon_{i,0}(k)$, *where* $P_{i,0}$ *can be also computed from* (4.11).

Then, comparing previous equation and (4.7), *it is clear that the gains* L_i *must be obtained according to* (4.9), *and the Lyapunov function* $V_{i,0}(k) = J^*_{i,0}(k) = \varepsilon_{i,0}(k)^\top P_{i,0}\varepsilon_{i,0}(k)$ *ensures the asymptotic convergence of* $\varepsilon_{i,0}$. □

Note that matrices $U_{i,\rho}$ weight the knowledge on the dynamics of the system whereas matrices $S_{i,\rho}$ weight the accuracy of the information provided by the local measurements of the system (when $\rho = 0$) and the information provided by the neighborhood (when $\rho > 0$). Thus, if the measurements are highly affected by noise, it will be reflected in $u_{i,0}$ and the weighting matrix $S_{i,\rho}$ must be designed consequently in order to not amplify the effect of the noises.

It is worth pointing out that the conditions established in Theorem 4.4.2 are completely local. That is, since matrices $W_{j,\rho-1}$ must be computed once in a initialization phase, no more information is required from the neighboring agents in order to design the observer gains.

4.4.2 Design feasibility

The existence of a matrix $P_{i,\rho}$ solution of the Algebraic Riccati equation stated in (4.11) is straightforward from the controllability of pair $(I_{n_{i,\rho}}, W^\top_{i,\rho}AW_{i,\rho})$ (see for instance [3]). It is a simple matter to check that regarding Popov-Belevitch-Hautus test (see, e.g., [26, Th.15.9]) the controllability is guaranteed due to the full rank of I_n matrix. Nevertheless, it left to prove the existence of gain matrices L_i and $N_{i,j,\rho}$ that fulfill expressions (4.9)-(4.10).

Theorem 4.4.3 *It is always possible, under Assumption* **??**, *to find a set of matrices L_i and $N_{i,j,\rho}$ satisfying equations* (4.9)-(4.10).

Proof 4.4.2 *The existence of L_i that fulfills* (4.9) *is a well-known consequence of observability of pair* $(C_iW_{i,0}, W_{i,0}^\top AW_{i,0})$ *that is equivalent to the observability of pair* $(C_iW_{i,0}, (S_{i,0} + P_{i,0})^{-1}P_{i,0}W_{i,0}^\top AW_{i,0})$ *since matrix* $(S_{i,0} + P_{i,0})^{-1}P_{i,0}$ *is a full rank matrix. Concerning gains $N_{i,j,\rho}$, equation* (4.10) *can be rewritten as*

$$(S_{i,\rho} + P_{i,\rho})^{-1}P_{i,\rho}W_{i,\rho}^\top AW_{i,\rho} = \bar{N}_{i,\rho}\Lambda_{i,\rho}W_{i,\rho},$$

where $\Lambda_{i,\rho} = \mathrm{col}(W_{j,\rho-1}^\top)_{j\in\mathcal{N}_i}$ *and* $\bar{N}_{i,\rho} = \mathrm{col}(N_{i,j,\rho}^\top)_{j\in\mathcal{N}_i}^\top$.

Hence, since matrix $(S_{i,\rho} + P_{i,\rho})^{-1}P_{i,\rho}$ *is a full rank matrix we only need to show that pair* $(\Lambda_{i,\rho}W_{i,\rho}, W_{i,\rho}^\top AW_{i,\rho})$ *is observable. From Popov-Belevitch-Hautus test the pair is observable if and only if:*

$$\mathit{rank}\begin{bmatrix} W_{i,\rho}^\top AW_{i,\rho} - \lambda I \\ \Lambda_{i,\rho}W_{i,\rho} \end{bmatrix} = n_{i,\rho}, \quad \forall\lambda \in \sigma(W_{i,\rho}^\top AW_{i,\rho}).$$

This condition can be guaranteed by only proving that rank$[\Lambda_{i,\rho}W_{i,\rho}] = n_{i,\rho}$. *Please note that according to Definition 2.3.2 and expression* (2.4), *the innovation obtained by agent i at hop ρ comes from the output matrices $C_{k,\rho}$ where $k \in \mathcal{N}_{i,\rho}$ and consequently, $k \in \mathcal{N}_{j,\rho-1}$ with $j \in \mathcal{N}_i$. Thus, it is easy to see that* $Im(W_{i,\rho}) \subseteq Im(\mathrm{col}(W_{j,\rho-1}^\top)_{j\in\mathcal{N}_i}^\top)$ *and consequently rank*$[\Lambda_{i,\rho}W_{i,\rho}] =$ *rank*$[W_{i,\rho}] = n_{i,\rho}$ *completing the proof.* $\square$

4.4.3 Stability analysis for the perturbed scenario

When norm-bounded disturbances noises are affecting the system and measurements, it is well-known that exponential stability can no longer be guaranteed. In the following result, it is stated that when the perturbed scenario is considered, the estimation error is globally uniformly ultimately bounded and, additionally, the bound is directly modulated with the energy of the exogenous signals.

Theorem 4.4.4 *Consider plant* (4.1) *observed by a set of agents that implements observation structure in* (4.3). *Then, under Assumption 4.2.1, if the observer gains are designed following Property 4.4.1, then the estimation error of the system is GUUB, i.e., the estimates are attracted and restricted to lay within a small region around the plant state.*

Proof 4.4.3 *Consider the following Lyapunov function for the transformed estimation error of agent i at hop ρ:*

$$V_{i,\rho}(k) = \varepsilon_{i,\rho}(k)^\top P_{i,\rho}\varepsilon_{i,\rho}(k),$$

with $P_{i,\rho}$ obtained from Theorem 1. The increment of the function is given by

$$\Delta V_{i,\rho}(k) = \varepsilon_{i,\rho}(k+1)^\top P_{i,\rho}\varepsilon_{i,\rho}(k+1) - \varepsilon_{i,\rho}(k)^\top P_{i,\rho}\varepsilon_{i,\rho}(k). \tag{4.17}$$

Using the dynamics of the transformed observation error when $\rho = 0$ given in (4.4), *it turns out:*

$$\begin{aligned}\Delta V_{i,0}(k) \;=\; & \varepsilon_{i,0}(k)^\top(E_{i,0}^\top P_{i,0}E_{i,0} - P_{i,0})\varepsilon_{i,0}(k) + w(k)^\top W_{i,0}P_{i,0}W_{i,0}^\top w(k) \\ + \; & n_i^\top(k)L_i^\top P_{i,0}L_i n_i(k) + 2\varepsilon_{i,0}(k)^\top E_{i,0}^\top P_{i,0}\left(W_{i,0}^\top w(k) + L_i n_i(k)\right), \\ + \; & 2n_i^\top(k)L_i^\top P_{i,0}W_{i,0}^\top w(k)\end{aligned} \tag{4.18}$$

where for simplicity in the notation we have denoted $E_{i,0} \triangleq W_{i,0}^\top AW_{i,0} - L_iC_iW_{i,0}$. From Theorem 4.4.2 we know that

$$\varepsilon_{i,0}(k)^\top(E_{i,0}^\top P_{i,0}E_{i,0} - P_{i,0})\varepsilon_{i,0}(k) = \varepsilon_{i,0}(k)^\top\tilde{\kappa}_{i,0}\varepsilon_{i,0}(k) = -\kappa_{i,0}(\varepsilon_{i,0}, u_{i,0}, k).$$

Using a consistent norm, equation (4.18) *can be bounded as*

$$\begin{aligned}\Delta V_{i,0}(k) \;\leq\; & -\lambda_{min}(\tilde{\kappa}_{i,0})||\varepsilon_{i,0}(k)||^2 + ||W_{i,0}P_{i,0}W_{i,0}||||w(k)||^2 \\ + \; & ||L_i^\top P_{i,0}L_i||||n_i(k)||^2 + 2||E_{i,0}^\top P_{i,0}W_{i,0}^\top||||\varepsilon_{i,0}(k)||||w(k)|| \\ + \; & 2||E_{i,0}^\top P_{i,0}L_i||||\varepsilon_{i,0}(k)||||n_i(k)|| + 2||L_i^\top P_{i,0}W_{i,0}^\top||||n_i(k)||||w(k)||.\end{aligned}$$

The right side of this equation is an algebraic second-order equation in $||\varepsilon_{i,\rho}(k)||$. This is, if we impose that the right side of the equation is equal to zero, then $\Delta V_{i,\rho}(k) < 0$:

$$a||\varepsilon_{i,0}(k)||^2 + b||\varepsilon_{i,0}(k)|| + c = 0, \tag{4.19}$$

where

$$\begin{aligned} a \;&= -\lambda_{min}(\tilde{\kappa}_{i,0}), \\ b \;&= 2||E_{i,0}^\top P_{i,0}W_{i,0}^\top||||w(k)|| + 2||E_{i,0}^\top P_{i,0}L_i||||n_i(k)||, \\ c \;&= ||W_{i,0}P_{i,0}W_{i,0}^\top||||w(k)||^2 + ||L_i^\top P_{i,0}L_i||||n_i(k)||^2 \\ &\quad +2||L_i^\top P_{i,0}W_{i,0}^\top||||n_i(k)||||w(k)||,\end{aligned}$$

and then, the unique positive root of the equation is given by

$$||\varepsilon_{i,0}(k)|| = f_{i,0}(||w(k)||, ||n_i(k)||),$$

where $f_{i,0}$ is a function that solves the second-order equation (4.19). *Thus, if we consider the extreme values of noise parameters, under Assumption 4.2.1, it is clear that $||\varepsilon_{i,0}(k)|| = f_{i,0}(\delta_w, \delta_{n_i})$ takes a finite value.*

Suppose any initial condition for the estimation error, in such a way that $||\varepsilon_{i,0}(k_0)|| < \infty$. Let us denote $\alpha_{i,0} := f_{i,0}(\delta_w, \delta_{n_i})$. Assume that $||\varepsilon_{i,0}(0)|| > \alpha_{i,0}$. In this case the Lyapunov function decreases and this will imply $||\varepsilon_{i,0}(k)|| < \alpha_{i,0}$ for some $k > k_0$. Since the one step evolution of the estimation error $\varepsilon_{i,0}(k+1)$ in (4.4) *is bounded provided that $||\varepsilon_{i,0}(k)|| < \alpha_{i,0}$, this finally proves that there exist a finite bound independent of time and initial conditions for $||\varepsilon_{i,0}(k)||$ for all $k > k_0$.*

From (4.5)*, it can be seen that the evolution of the estimation error of agent i at hop ρ depends on the estimation error of that agent at the previous hops and the estimation error of the neighborhoods at hop $\rho - 1$, thus revealing a cascade structure. Hence, if we apply the same procedure recursively from $\rho = 0$ to $\rho = \ell_i$, we can reach to an algebraic second-order equation in $||\varepsilon_{i,\rho}(k)||$ whose coefficients depend on the solution of the second-order equations in the transformed estimation error of agent i and its neighborhood $j \in \mathcal{N}_i$ at the previous hop. It is clear that the solution of this equation is finite completing the proof.* □

Corollary 4.4.5 *If the measurements of the agents are not affected by noise, the infinity norm of the local transformed estimation error is decreasing as long as*

$$||\varepsilon_{i,0}(k)|| > \mu_{i,0}\delta_w,$$

where

$$\mu_{i,0} = \frac{||E_{i,0}^\top P_{i,0} W_{i,0}^\top|| + \sqrt{\left(||E_{i,0}^\top P_{i,0} W_{i,0}^\top||^2 + \lambda_{min}(\kappa_{i,0})||W_{i,0} P_{i,0} W_{i,0}^\top||\right)}}{\lambda_{min}(\tilde{\kappa}_{i,0})}.$$

Proof 4.4.4 *The proof is based on solving* (4.19) *when* $||n_i(k)|| = 0$. □

4.4.4 Tuning procedure proposed

In this section, a general method to design the weighting matrices $U_{i,\rho}$ and $S_{i,\rho}$ is presented. The aim of the method is to design the weighting matrices to each agent $i \in \mathcal{V}$ according to the distance between agent i and the agent which constitutes the source of the information used by i to reconstruct the unobservable subspace.

Additionally, a scalar parameter γ_i is introduced in order to allow the user to change the proportionality between matrices $U_{i,\rho}$ and $S_{i,\rho}$ regarding their experience with the process. Thus, this chapter proposes the following values for the weighting matrices:

$$U_{i,\rho} = \gamma_i I_{n_{i,\rho}}, \tag{4.20}$$

$$S_{i,\rho} = 10^{\rho+1} I_{n_{i,\rho}}, \tag{4.21}$$

where $\gamma_i \in \mathbb{R}$. Thus, the design of the observer has been reduced to a problem in which it is only necessary to fix the value of one scalar.

If weighting matrices $U_{i,\rho}$ and $S_{i,\rho}$ are designed following equations (4.20), (4.21) and $\gamma_i = 1$ for all $i \in \mathcal{V}$, the matrices are chosen in order to weight the relative distance to the agent. Note that the design proposed in (4.20)-(4.21) gives a higher value to matrix $S_{i,\rho}$ as long as the distance between agent i and the agent which constitutes the source of the information increments. In this way, a more aggressive feedback signal is imposed to the local corrections and it is becoming softer as ρ arises.

4.5 Simulation results

In order to show the robustness of the distributed design of the observer some simulation examples are driven in this section. Consider the following system where there is one state with a stable dynamics, a pair of conjugated imaginary poles and a state with an unstable dynamic:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^+ = \begin{bmatrix} 0.95 & 0 & 0 & 0 \\ 0 & 0.8606 & -1.3368 & 0 \\ 0 & 0.0941 & 0.9315 & 0 \\ 0 & 0 & 0 & 1.015 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

and is observed by a set of four agents ($y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3,\ y_4 = x_4$) with the network topology defined in Figure 4.1.

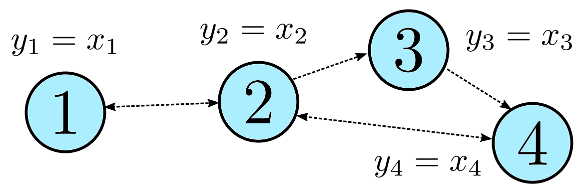

Figure 4.1: Network topology considered.

The basis vectors of the observable subspace for each agent can be easily obtained as:

$$W_{1,0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad W_{2,0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad W_{3,0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad W_{4,0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that agents 2 and 3 have the same observable subspace and, therefore, they will estimate states x_2 and x_3 based only on their local measurements.

Example 4.5.1 *In this example the performance of the estimation is shown. According to Assumption 4.2.1, consider that the infinity norm of the noise terms $w(k)$ and $n_i(k)$ for all $i \in \mathcal{V}$ and for all time k are upper-bounded and the bounds are given by:*

$$\delta_w = 0, \quad \delta_{n_1} = 0.8, \quad \delta_{n_2} = 0.9, \quad \delta_{n_3} = 0.7, \quad \delta_{n_4} = 0.6.$$

Consider a value of $\gamma_i = 1$ for all $i \in \mathcal{V}$. In Figure 4.2 the evolution of the estimation error for agent 4, e_4, is shown. It is worth pointing out that the estimation error of state x_4, which belongs to the observable subspace of the agent, decreases drastically achieving a short convergence time. Errors $e_4(2)$ and $e_4(3)$, that according to the graph correspond to the innovation introduced at hop $\rho = 1$, start decreasing when $e_4(1)$ reaches the steady state. Lastly, state x_4, has a convergence rate slower than the others due to the fact that this state belongs to the innovation at $\rho = 2$. Thus, it is easy to see the cascade structure of the observer.

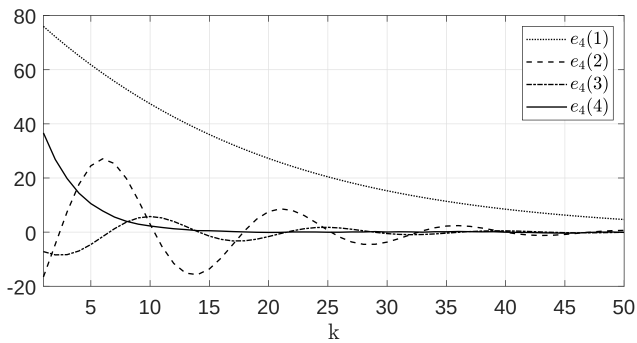

Figure 4.2: Estimation error evolution for agent 4.

Figure 4.3 depicts the evolution of maximum value of the $||e_i(k)||_\infty$ for every agent $i \in \mathcal{V}$ and for the different observers modifying the value of γ_i. Note that, for *high values of γ_i for all $i \in \mathcal{V}$, the feedback action is more aggressive achieving a lower settling time than when the value of γ_i increases. However, the noise rejection for low values of γ_i work better in the steady state. Recall that, for $\gamma_i = 10$, the observers trade off between a good convergence rate and a good noise rejection in steady state, achieving a good performance according to both parameters.*

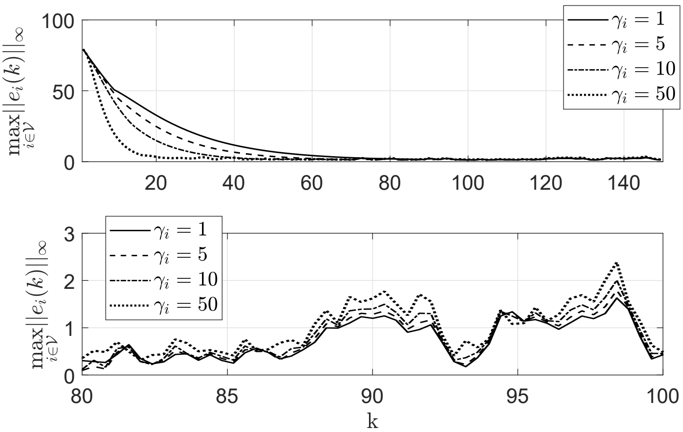

Figure 4.3: Evolution of the maximum value of the $||e_i(k)||_\infty$ for every agent $i \in \mathcal{V}$ according to γ_i.

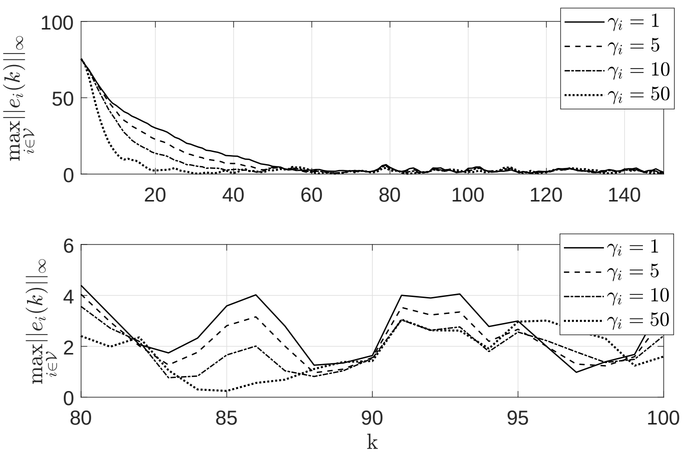

Figure 4.4: Evolution of the maximum value of the $||e_i(k)||_\infty$ for every agent $i \in \mathcal{V}$ according to γ_i.

Example 4.5.2 *In this example we will consider the same system and network topology than in Example 4.5.1 but this time, let us consider the sequel noise bounds:*

$$\delta_w = 0.6, \quad \delta_{n_i} = 0, \quad \forall i \in \mathcal{V}.$$

Figure 4.4 depicts the evolution of the maximum value of $||e_i(k)||_\infty$ *for every agent* $i \in \mathcal{V}$ *for different values of* γ_i*. Note that, in this second example, the noises are only introduced in the system dynamics. This, yields to a situation in which if* $U_{i,\rho}$ *is greater than* $S_{i,\rho}$*, the cost function* (4.6) *is weighting higher the estimation error, relying more on the measurements taken by the agents than on the system model. Thus, for higher values of* γ_i *a better performance in steady state is obtained.*

4.6 Conclusions

By using a novel observer structure, a distributed LQ-based design has been introduced in which, adjusting some weighting matrices, the performance of the estimation can be tuned. A method to tune these weighting matrices has been presented in such a way that it is only necessary to adjust the value of a parameter γ_i. The stability of the presented observer structure has been proven under the unperturbed and perturbed scenario. Some simulation examples have been introduced in order to show the effectiveness of the algorithm.

Chapter 5

A data-fusion-based approach for state estimation

5.1 Introduction

This chapter deals with the problem of estimating the state of a perturbed plant by a network of agents executing a distributed data-fusion-based algorithm. Each agent knows the model of the plant and can measure some system outputs. The rest of the necessary outputs are obtained through the exchange of information with neighboring agents. This chapter considers that the agents communicate through a multi-hop network, where data transmitted may take several sampling instants to reach its final destination. In other words, the communications are affected by graph-induced delays. Conversely to other approaches such as [73], we are not considering signal transmission delays or signal processing delays in the agents. The proposed observer structure decouples the state-space into several subspaces according to the observable modes considering the information received at each sample time. This exploits the idea that was previously presented by the authors in [45], where the number of agents was limited to two, and in [13], where a generalization to an arbitrary number of agents was adopted. However, neither of those chapters considered the data-fusion scheme. The observer gains are designed in order to guarantee the stability of the distributed observer in spite of the presence of delays. The main contributions of the chapter are listed next:

- The introduction of a novel data-fusion-based observer structure able to be designed in a distributed fashion.
- Unlike the conventional data fusion approaches [35] or [9], the information is not required to spread through the network in a single sample time, thereby

relaxing the requirements of the network.

- The observer design reduces the exchange of information with respect to other data fusion algorithms, due to the fact that the agents are not required to collect the information of every agent to reconstruct the whole state.
- In case of duplicated information, the proposed subspace decomposition allows the observer to be selective when deciding the agents who will be the source of the required data.

5.2 Problem formulation

Consider a set of agents $\mathcal{V} = \{1, 2, \ldots, p\}$ connected according to a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and intended to distributedly estimate the state of the following discrete-time LTI system:

$$x(k+1) = Ax(k), \tag{5.1}$$
$$y_i(k) = C_i x(k). \tag{5.2}$$

We assume that the information flows slowly through the network, consuming one sample time to get from any agent to its neighbors. This is the case, for instance, of networks set up based on Zigbee [40].

According to the network connectivity of the agents, Assumption 2.3.4 is a necessary assumption.

5.3 Observer structure and design goal

This section presents the observer structure that makes use of the notions previously introduced in Chapter 2:

$$\begin{aligned}\hat{x}_i(k+1) = &\underbrace{A\hat{x}_i(k)}_{(a)} \\ &+ \underbrace{\sum_{\rho=0}^{\ell_i} W_{i,\rho} N_{i,\rho}(k) W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top \left(y_j(k-\rho) - C_j\hat{x}_i(k-\rho)\right)}_{(b)},\end{aligned} \tag{5.3}$$

where ρ is the distance from agent i to the agent that constitutes the source of information, and $N_{i,\rho} \in \mathbb{R}^{n_{i,\rho}\times n_{i,\rho}}$ are a set of gains to be designed. Recall that, since we are assuming slow communication networks in which the information consumes one sample time to get from one agent to its neighbors, this information reaches the destination agent with a fixed delay of ρ sample times (The fixed delay for the communication from agent j to agent i depends on their relative position in the graph. Therefore we could have written $\rho_{i,j}$ instead of ρ. For the sake of simplicity on the notation, we have kept the second option.). In other words, there exists a match between the total delay the packet suffers between sender and receiver, and the hop at which the information affects (It is trivially easy to modify the delay for the communications to other values higher than one. However, this would make the notation harder.).

The observer structure proposed in (5.3) decomposes the observer dynamics in two different terms:

(a) The first term is the classical model-based open-loop prediction term.
(b) The second one is a correction term. The agents belonging to the $\rho-$ hop reachable set of i, $\mathcal{N}_{i,\rho}$, communicate the measurements made to agent i. Since transmitted measurements flow through the graph at a rate of 1 hop per sampling time, any agent i receives the measurements from its ρ-hop reachable set with a constant delay of ρ sampling times. The correction made with these measurements is projected into $Im(W_{i,\rho})$ and multiplied by the gain matrix $N_{i,\rho}(k)$. The result is used as weights to perform linear combinations of $W_{i,\rho}$. Thus, these corrections only affect the observable subspace generated by the innovations of the neighborhood of agent i at hop ρ.

Remark 5.3.1 *It is worth pointing out that if $\ell_i = 0$, pair (C_i, A) is observable and consequently $W_{i,0}$ is a full rank matrix. Thus, Equation (5.3) can be rewritten as:*

$$\hat{x}_i(k+1) = A\hat{x}_i(k) + L_i(k)(y_i(k) - C_i\hat{x}_i(k)),$$

where $L_i(k) = W_{i,0}N_{i,0}(k)W_{i,0}^\top C_i^\top$. Note that above expression is clearly the well-known Luenberger observer structure.

The goal of this chapter is to design the gain matrices $N_{i,\rho}(k)$ in structure (5.3) for every agent i and every hop $\rho \in \{0, \dots, \ell_i\}$ to solve the following problem:

Problem 5.3.1 *(Distributed data fusion) Given plant (5.1) and (5.2), and the interconnection graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, design gains $N_{i,\rho}(k)$ in (5.3) such that all the estimates $\hat{x}_i$ asymptotically converge to the actual plant state x.*

The solution for Problem 5.3.1 is introduced in Section 5.5. However, prior to that, it is necessary to introduce some properties on which subsequent developments are supported.

5.3.1 Distributed observer setup

It is worth mentioning that observer structure (5.3) needs some neighboring information before starting the estimation phase. That is, the construction of matrices $W_{i,\rho}$ for every $\rho \in \{0, \ldots, \ell_i\}$ and the value of ℓ_i for every agent $i \in \mathcal{V}$ it is needed before starting the estimation procedure.

This section presents an algorithm to design the matrices and parameters aforementioned. Note that although gain matrices $N_{i,\rho}(k)$ are also required before the estimation phase, this design is tackled in the following sections. The pseudocode of the setup algorithm is given next:

- For every agent i do:
  - Set $\rho = 0$.
  - Perform the two steps:
    - Exchange matrices $C_{i,\rho}$ with the neighborhood $\mathcal{N}_i$.
    - Compute $C_{i,\rho+1}$ and matrix $W_{i,\rho}$. Include the path in the routing tables of the interconnection nodes.
  - If pair $(C_{i,\rho+1}, A)$ is detectable, then stop and fix $\ell_i = \rho+1$. Otherwise increment ρ and perform the two above steps again.

The routing tables of the interconnection nodes includes the information that the agents must know in order to route the information from one source agent to the destination agent in a direct path.

After this initialization phase, that finishes after a finite number of steps, the agents are ready to execute their estimation of the state in a distributed way.

Remark 5.3.2 *By letting this algorithm be executed successive times, and not just at the initialization phase, the agents can detect changes in the topology and, then, redesign their observer gains. Hence, the proposed observer can be made resilient to time-varying topologies.*

Remark 5.3.3 *Consider a situation in which the destination agent i is selective with the agent who will act as a source of information to reconstruct some part of the state. In this situation, agent i must take this fact under consideration in the setup algorithm, modifying matrices $C_{i,\rho}$ and consequently matrices $W_{i,\rho}$ (which denote the innovation basis of agent i of the observable subspace at hop ρ).*

5.4 Estimation error dynamics

This section presents the evolution of the transformed estimation error, $\varepsilon_{i,\rho}(k)$. After introducing these dynamics, it will become clear that new delayed versions of this error will need to be defined, together with their associate dynamics.

Proposition 5.4.1 *Consider the network of agents described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where every agent i implements the observer structure* (5.3) *to estimate the state of the system* (5.1). *Then, the transformed estimation error dynamics at every hop ρ is given by the following equation:*

$$\varepsilon_{i,\rho}(k+1) = W_{i,\rho}^{\top} A \sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r}(k) \\ - N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} C_j \sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r}(k-\rho), \quad (5.4)$$

for all $\rho = \{0, \ldots, \ell_i\}$.

Proof 5.4.1 *Let us write first the evolution of the estimation error dynamics for system* (5.1) *under the observation structure in* (5.3)*:*

$$e_i(k+1) = x(k+1) - \hat{x}_i(k+1) = \\ Ae_i(k) - \sum_{\rho=0}^{\ell_i} W_{i,\rho} N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} C_j e_i(k-\rho).$$

Using (2.10), *we can write the transformed estimation error dynamics for agent i at hop ρ:*

$$\varepsilon_{i,\rho}(k+1) = W_{i,\rho}^{\top} e_i(k+1) = W_{i,\rho}^{\top} A e_i(k) \\ - N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} C_j e_i(k-\rho),$$

where $W_{i,\rho}^{\top} W_{i,\rho} = I_{n_{i,\rho}}$ and Lemma 2.3.1 (i) has been used. Next, relying on Equation (2.11), *we know the expression of e_i in $\varepsilon_{i,\rho}$ coordinates, so previous equation can be accordingly modified:*

$$\varepsilon_{i,\rho}(k+1) = W_{i,\rho}^{\top} A \left(\bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1}(k) + \sum_{r=0}^{\ell_i} W_{i,r}\varepsilon_{i,r}(k) \right) \\ - N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} C_j \left(\bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1}(k-\rho) + \sum_{r=0}^{\ell_i} W_{i,r}\varepsilon_{i,r}(k-\rho) \right).$$

From Lemma 2.3.1 (iv) it holds $W_{i,\rho}^\top A\left(\bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1}+\sum_{r=\rho+1}^{\ell_i}W_{i,r}\varepsilon_{i,r}\right)=0$ *so we can rewrite the above equation as:*

$$\begin{aligned}\varepsilon_{i,\rho}(k+1) &= W_{i,\rho}^\top A\sum_{r=0}^{\rho}W_{i,r}\varepsilon_{i,r}(k)-N_{i,\rho}(k)W_{i,\rho}^\top \\ &\quad\times\sum_{j\in\mathcal{N}_{i,\rho}}C_j^\top\left(C_j\bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1}(k-\rho)+C_j\sum_{r=0}^{\ell_i}W_{i,r}\varepsilon_{i,r}(k-\rho)\right).\end{aligned}$$

Next, from Lemma 2.3.2 we know that $C_jW_{i,\rho'}=0$ *for all* $j\in\mathcal{N}_{i,\rho}$ *with* $\rho'>\rho$ *and therefore*

$$\begin{aligned}\varepsilon_{i,\rho}(k+1) &= W_{i,\rho}^\top A\sum_{r=0}^{\rho}W_{i,r}\varepsilon_{i,r}(k) \\ &\qquad -N_{i,\rho}(k)W_{i,\rho}^\top\sum_{j\in\mathcal{N}_{i,\rho}}C_j^\top C_j\sum_{r=0}^{\rho}W_{i,r}\varepsilon_{i,r}(k-\rho),\end{aligned}$$

which is the desired expression. □

Now, let us define the delayed transformed estimation error of agent i at hop ρ, $\bar{\varepsilon}_{i,\rho}$, as the vector that stacks the transformed estimation error of agent i at hop ρ from time k to $k-\ell_i$, that is:

$$\bar{\varepsilon}_{i,\rho}(k) := \begin{bmatrix}\varepsilon_{i,\rho}(k)\\ \varepsilon_{i,\rho}(k-1)\\ \vdots\\ \varepsilon_{i,\rho}(k-\ell_i)\end{bmatrix}, \tag{5.5}$$

and let us define $\bar{\varepsilon}_i$ as the vector that stacks the delayed transformation error of agent i at every hop $\rho\in\{0,\dots,\ell_i\}$ sorted in decreasing order

$$\bar{\varepsilon}_i(k) := \begin{bmatrix}\bar{\varepsilon}_{i,\ell_i}(k)\\ \bar{\varepsilon}_{i,\ell_i-1}(k)\\ \vdots\\ \bar{\varepsilon}_{i,0}(k)\end{bmatrix}. \tag{5.6}$$

Observe that $\varepsilon_{i,\rho}(k)=S_{i,\rho}\bar{\varepsilon}_{i,\rho}(k)$, with

$$S_{i,\rho}=\left[I_{n_{i,\rho}}\ 0\dots0\right]\in\mathbb{R}^{n_{i,\rho}\times(\ell_i+1)n_{i,\rho}}.$$

Analogously, $\varepsilon_{i,\rho}(k) = \bar{S}_{i,\rho}\bar{\varepsilon}_i(k)$, with

$$\bar{S}_{i,\rho} = S_{i,\rho}\begin{bmatrix}0 & \cdots & 0 & I_{(\ell_i+1)n_{i,\rho}} & 0 & \cdots & 0\end{bmatrix} \in \mathbb{R}^{n_{i,\rho}\times\sum_{r=0}^{\ell_i}(\ell_i+1)n_{i,r}}.$$

Based on these error vectors, we can obtain an expression for the dynamics of the delayed transformed estimation error, $\bar{\varepsilon}_{i,\rho}$, that will be useful later.

Proposition 5.4.2 *Consider the network of agents described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where every agent i implements the observer structure* (5.3) *to estimate the state of the system* (5.1). *Then, the delayed transformed estimation error dynamics at hop ρ is given by the following equation:*

$$\bar{\varepsilon}_{i,\rho}(k+1) = \sum_{r=0}^{\rho} \triangle_{i,(\rho,r)}(k)\bar{\varepsilon}_{i,r}(k) \tag{5.7}$$

being $\triangle_{i,(\rho,\rho)}(k) \in \mathbb{R}^{(\ell_i+1)(n_{i,\rho}\times n_{i,\rho})}$:

$$\triangle_{i,(\rho,\rho)}(k) = \tag{5.8}$$
$$\begin{bmatrix} W_{i,\rho}^\top A W_{i,\rho} & 0 & \cdots & -N_{i,\rho}(k)W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top C_j W_{i,\rho} & \cdots & 0 & 0 \\ I_{n_{i,\rho}} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & I_{n_{i,\rho}} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I_{n_{i,\rho}} & 0 \end{bmatrix},$$

and $\triangle_{i,(\rho,r)}(k) \in \mathbb{R}^{(\ell_i+1)(n_{i,\rho}\times n_{i,r})}$:

$$\triangle_{i,(\rho,r)}(k) = \tag{5.9}$$
$$\begin{bmatrix} W_{i,\rho}^\top A W_{i,r} & 0 & \cdots & -N_{i,\rho}(k)W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top C_j W_{i,r} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{bmatrix},$$

for $r \neq \rho$ and $r, \rho \in \{0, \ldots, \ell_i\}$ where the terms $-N_{i,\rho}(k)W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top C_j W_{i,\rho}$ and $-N_{i,\rho}(k)W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top C_j W_{i,r}$ are placed in the block component $(1, \rho+1)$ of the matrices $\triangle_{i,(\rho,\rho)}$ and $\triangle_{i,(\rho,r)}$, respectively.

The proof is immediate by substituting Equation (5.4) into Equation (5.5).

5.5 Distributed data fusion

This section tackles Problem 5.3.1. In particular, the section presents necessary and sufficient stability conditions for the observer structure (5.3) in absence of perturbations. Moreover, an LMI method (see [7] for details) for the design of gain matrices $N_{i,\rho}(k)$ that guarantees stability is introduced. This design is independent of time and then, for simplicity in the notation we will use $N_{i,\rho} = N_{i,\rho}(k)$.

Theorem 5.5.1 *Consider the network of agents described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where every agent i implements the observer structure* (5.3) *to estimate the state of the system* (5.1). *Then, the estimates of all the agents tend asymptotically to the actual plant state if and only if matrices $\triangle_{i,(\rho,\rho)}$ are Schur for every $\rho \in \{0, \dots, \ell_i\}$.*

Proof 5.5.1 *The dynamics of the delayed transformed estimation error of agent i at hop ρ is given in Proposition 5.4.2. By stacking the transformed estimation error of agent i at every hop the following expression can be obtained:*

$$\underbrace{\begin{bmatrix} \bar{\varepsilon}_{i,\ell_i}(k+1) \\ \bar{\varepsilon}_{i,\ell_i-1}(k+1) \\ \vdots \\ \bar{\varepsilon}_{i,0}(k+1) \end{bmatrix}}_{\bar{\varepsilon}_i(k+1)} = \begin{bmatrix} \triangle_{i,(\ell_i,\ell_i)} & \triangle_{i,(\ell_i,\ell_i-1)} & \cdots & \triangle_{i,(\ell_i,0)} \\ 0 & \triangle_{i,(\ell_i-1,\ell_i-1)} & \cdots & \triangle_{i,(\ell_i-1,0)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \triangle_{i,(0,0)} \end{bmatrix} \underbrace{\begin{bmatrix} \bar{\varepsilon}_{i,\ell_i}(k) \\ \bar{\varepsilon}_{i,\ell_i-1}(k) \\ \vdots \\ \bar{\varepsilon}_{i,0}(k) \end{bmatrix}}_{\bar{\varepsilon}_i(k)}. \tag{5.10}$$

Please note that (5.10) *reveals a cascade structure in which the delayed transformed estimation error at each hop ρ depends on the errors at previous hops. Thus, the eigenvalues of the block upper triangular matrix in* (5.10) *are given by the eigenvalues of the corresponding matrices placed in its diagonal, which are the matrices defined in* (5.8) *establishing the proof.* □

It is worth pointing out that the stability of the distributed observer can be seen as the stability of several constant time-delay discrete-time systems. Next, a design method for gain matrices $N_{i,\rho}$ that met Theorem 5.5.1 is presented.

Theorem 5.5.2 *(Design for stability) If there exist matrices $R_{i,\rho} \in \mathbb{R}^{n_{i,\rho}\times n_{i,\rho}}$, $\mathcal{X}_{i,\rho} \in \mathbb{R}^{n_{i,\rho}\times n_{i,\rho}}$ and $\mathcal{Y}_{i,\rho} = \begin{bmatrix} \mathcal{Y}_{i,\rho}^{(1,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\ell_i)} \\ \vdots & \ddots & \vdots \\ \mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} \end{bmatrix} \in \mathbb{R}^{\ell_i(n_{i,\rho}\times n_{i,\rho})}$ such that the LMI*

$$
\left[\begin{array}{cccccc|c}
\mathcal{X}_{i,\rho} - \mathcal{Y}_{i,\rho}^{(1,1)} & -\mathcal{Y}_{i,\rho}^{(1,2)} & \cdots & -\mathcal{Y}_{i,\rho}^{(1,\rho+1)} & \cdots & -\mathcal{Y}_{i,\rho}^{(1,\ell_i)} & 0 & W_{i,\rho}^{\top} A^{\top} W_{i,\rho}\mathcal{X}_{i,\rho} \\
-\mathcal{Y}_{i,\rho}^{(2,1)} & \mathcal{Y}_{i,\rho}^{(1,1)} - \mathcal{Y}_{i,\rho}^{(2,2)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\rho)} - \mathcal{Y}_{i,\rho}^{(2,\rho+1)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\ell_i-1)} - \mathcal{Y}_{i,\rho}^{(2,\ell_i)} & \mathcal{Y}_{i,\rho}^{(1,\ell_i)} & 0 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
-\mathcal{Y}_{i,\rho}^{(\rho+1,1)} & \mathcal{Y}_{i,\rho}^{(\rho,1)} - \mathcal{Y}_{i,\rho}^{(\rho+1,2)} & \cdots & \mathcal{Y}_{i,\rho}^{(\rho,\rho)} - \mathcal{Y}_{i,\rho}^{(\rho+1,\rho+1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\rho,\ell_i-1)} - \mathcal{Y}_{i,\rho}^{(\rho+1,\ell_i)} & \mathcal{Y}_{i,\rho}^{(\rho,\ell_i)} & -W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} C_j W_{i,\rho} R_{i,\rho}^{\top} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
-\mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \mathcal{Y}_{i,\rho}^{(\ell_i-1,1)} - \mathcal{Y}_{i,\rho}^{(\ell_i,2)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i-1,\rho)} - \mathcal{Y}_{i,\rho}^{(\ell_i,\rho+1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i-1)} - \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} & \mathcal{Y}_{i,\rho}^{(\ell_i-1,\ell_i)} & 0 \\
0 & \mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\rho)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i-1)} & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} & 0 \\
\hline
\mathcal{X}_{i,\rho} W_{i,\rho}^{\top} A W_{i,\rho} & 0 & \cdots & -R_{i,\rho} W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} C_j W_{i,\rho} & \cdots & 0 & 0 & \mathcal{X}_{i,\rho}
\end{array}\right] > 0, \tag{5.11}
$$

is satisfied, then gain $N_{i,\rho} = R_{i,\rho}\mathcal{X}_{i,\rho}^{-1}$ *stabilizes matrices* (5.8), *and therefore the estimates of every agent tends asymptotically to the actual plant state.*

Proof 5.5.2 *From Lyapunov theory [7], Theorem 5.5.1 is fulfilled if and only if there exists a positive definite matrix* $P_{i,\rho} \in \mathbb{R}^{(\ell_i+1)(n_{i,\rho}\times n_{i,\rho})}$ *such that:*

$$
\begin{array}{l}
P_{i,\rho} > 0, \\
P_{i,\rho} - \triangle_{i,(\rho,\rho)}^{\top} P_{i,\rho} \triangle_{i,(\rho,\rho)} > 0.
\end{array}
$$

Now, define $P_{i,\rho} = \begin{bmatrix} \mathcal{X}_{i,\rho} & 0 \\ 0 & \mathcal{Y}_{i,\rho} \end{bmatrix}$ *with* $\mathcal{X}_{i,\rho} \in \mathbb{R}^{n_{i,\rho}\times n_{i,\rho}}$ *and* $\mathcal{Y}_{i,\rho} \in \mathbb{R}^{\ell_i(n_{i,\rho}\times n_{i,\rho})}$.

With some mathematical manipulations, previous conditions can be rewritten as

$$
\begin{aligned}
0 < {} & \begin{bmatrix} \mathcal{X}_{i,\rho} & 0 & \cdots & 0 \\ 0 & \mathcal{Y}_{i,\rho}^{(1,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\ell_i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} \end{bmatrix} - \begin{bmatrix} W_{i,\rho}^{\top} A^{\top} W_{i,\rho} & I_{n_{i,\rho}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -W_{i,\rho}^{\top} M_{i,\rho}^{\top} W_{i,\rho} N_{i,\rho}^{\top} & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_{n_{i,\rho}} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
& \times \begin{bmatrix} \mathcal{X}_{i,\rho} & 0 & \cdots & 0 \\ 0 & \mathcal{Y}_{i,\rho}^{(1,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\ell_i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} \end{bmatrix} \\
& \times \begin{bmatrix} W_{i,\rho}^{\top} A W_{i,\rho} & 0 & \cdots & -N_{i,\rho} W_{i,\rho}^{\top} M_{i,\rho} W_{i,\rho} & \cdots & 0 & 0 \\ I_{n_{i,\rho}} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & I_{n_{i,\rho}} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I_{n_{i,\rho}} & 0 \end{bmatrix}.
\end{aligned}
$$

Then, after some manipulations, the following inequality is obtained:

$$
\begin{aligned}
0 < {} & \begin{bmatrix} \mathcal{X}_{i,\rho} & 0 & \cdots & 0 \\ 0 & \mathcal{Y}_{i,\rho}^{(1,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\ell_i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} \end{bmatrix} - \begin{bmatrix} \mathcal{Y}_{i,\rho}^{(1,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\ell_i)} & 0 \\ \vdots & \ddots & \vdots & 0 \\ \mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \\
& - \begin{bmatrix} W_{i,\rho}^{\top} A^{\top} W_{i,\rho} \\ 0 \\ \vdots \\ -W_{i,\rho}^{\top} \sum\limits_{j \in \mathcal{N}_{i,\rho}} C_j^{\top} C_j W_{i,\rho} N_{i,\rho}^{\top} \\ \vdots \\ 0 \end{bmatrix} \mathcal{X}_{i,\rho} \\
& \times \begin{bmatrix} W_{i,\rho}^{\top} A W_{i,\rho} & 0 & \cdots & -N_{i,\rho} W_{i,\rho}^{\top} \sum\limits_{j \in \mathcal{N}_{i,\rho}} C_j^{\top} C_j W_{i,\rho} & \cdots & 0 \end{bmatrix}.
\end{aligned}
$$

Adding the first two matrices and applying the Schur complement [7], previous *inequality is equivalent to*

$$
\left[\begin{array}{cccccccc|c}
\mathcal{X}_{i,\rho}-\mathcal{Y}_{i,\rho}^{(1,1)} & -\mathcal{Y}_{i,\rho}^{(1,2)} & \cdots & -\mathcal{Y}_{i,\rho}^{(1,\rho+1)} & \cdots & -\mathcal{Y}_{i,\rho}^{(1,\ell_i)} & 0 & & W_{i,\rho}^{\top}A^{\top}W_{i,\rho} \\
-\mathcal{Y}_{i,\rho}^{(2,1)} & \mathcal{Y}_{i,\rho}^{(1,1)}-\mathcal{Y}_{i,\rho}^{(2,2)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\rho)}-\mathcal{Y}_{i,\rho}^{(2,\rho+1)} & \cdots & \mathcal{Y}_{i,\rho}^{(1,\ell_i-1)}-\mathcal{Y}_{i,\rho}^{(2,\ell_i)} & \mathcal{Y}_{i,\rho}^{(1,\ell_i)} & & 0 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
-\mathcal{Y}_{i,\rho}^{(\rho+1,1)} & \mathcal{Y}_{i,\rho}^{(\rho,1)}-\mathcal{Y}_{i,\rho}^{(\rho+1,2)} & \cdots & \mathcal{Y}_{i,\rho}^{(\rho,\rho)}-\mathcal{Y}_{i,\rho}^{(\rho+1,\rho+1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\rho,\ell_i-1)}-\mathcal{Y}_{i,\rho}^{(\rho+1,\ell_i)} & \mathcal{Y}_{i,\rho}^{(\rho,\ell_i)} & & -W_{i,\rho}^{\top}\sum_{j\in\mathcal{N}_{i,\rho}}C_j^{\top}C_jW_{i,\rho}N_{i,\rho}^{\top} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
-\mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \mathcal{Y}_{i,\rho}^{(\ell_i-1,1)}-\mathcal{Y}_{i,\rho}^{(\ell_i,2)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i-1,\rho)}-\mathcal{Y}_{i,\rho}^{(\ell_i,\rho+1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i-1)}-\mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} & \mathcal{Y}_{i,\rho}^{(\ell_i-1,\ell_i)} & & 0 \\
0 & \mathcal{Y}_{i,\rho}^{(\ell_i,1)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\rho)} & \cdots & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i-1)} & \mathcal{Y}_{i,\rho}^{(\ell_i,\ell_i)} & & 0 \\
\hline
W_{i,\rho}^{\top}AW_{i,\rho} & 0 & \cdots & -N_{i,\rho}W_{i,\rho}^{\top}\sum_{j\in\mathcal{N}_{i,\rho}}C_j^{\top}C_jW_{i,\rho} & \cdots & 0 & 0 & & \mathcal{X}_{i,\rho}^{-1}
\end{array}\right] > 0.
$$

Note that the obtained matrix inequality is not an LMI because there are terms on $\mathcal{X}_{i,\rho}$ *and* $\mathcal{X}_{i,\rho}^{-1}$*. If the matrix inequality is pre-multiplied and post-multiplied by the symmetric, non-singular matrix* $\begin{bmatrix} I_{(\ell_i+1)n_{i,\rho}} & 0 \\ 0 & \mathcal{X}_{i,\rho} \end{bmatrix}$*, and by using the change of variables* $R_{i,\rho} = N_{i,\rho}\mathcal{X}_{i,\rho}$*, LMI* (5.11) *is obtained, establishing the proof.* □

Recall that LMI (5.11) can be solved locally by every agent, requiring just the information available after the setup described in Section 5.3.1. The choice of a block-diagonal Lyapunov matrix introduces conservatism, but it is required to transform the matrix inequality into an LMI. Otherwise, the previous condition would be necessary and sufficient for stabilization.

5.6 Simulation results

In order to show the stability of the distributed observer some simulations are driven in this section. Consider the following system matrix:

$$A = \begin{bmatrix} 0.95 & 0 & 0 & 0 \\ 0 & 0.8606 & -1.3368 & 0 \\ 0 & 0.0941 & 0.9315 & 0 \\ 0 & 0 & 0 & 1.015 \end{bmatrix}. \tag{5.12}$$

Note that in this example we are considering four states (one stable, one unstable and two conjugated imaginary poles).

The system is observed by a set of three agents ($y_1 = x_1, y_2 = x_3, y_3 = x_4$) with the network topology: $1 \longleftrightarrow 2 \longleftrightarrow 3$.

If we run the LMI presented in Theorem 5.5.2, the gain matrices $N_{i,\rho}$ exposed in Table 5.1 are obtained.

ρ	Agent 1	Agent 2	Agent 3
0	0.6010	$\begin{bmatrix} 0 & -1.0434 \\ 0 & 0.5485 \end{bmatrix}$	0.6114
1	$\begin{bmatrix} 0.5935 & 0 \\ -1.9165 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.6868 & 0 \\ 0.6017 & 0 \end{bmatrix}$	0.6114
2	0.4180	-	0.3405

Table 5.1: Gain matrices $N_{i,\rho}$ obtained after solving LMI (5.11).

Figure 5.1 depicts the evolution of the system state and agent 1 estimates (in dashed lines). We have used Matlab to solve the LMI (5.11) obtaining a feasible solution. It is worth pointing out that the solution provided by the solver does not optimize a specific cost function, so then, we are not actuating over the estimation performance.

5.7 Conclusions

This chapter has presented a novel observer structure based on data fusion. Relaying in the multi-hop subspace decomposition presented in Chapter 2, it is possible to decouple the modes according to the observability of pair $(C_{i,\rho}, A)$ for the different hops ρ. This fact makes possible to introduce an observer design that only requires

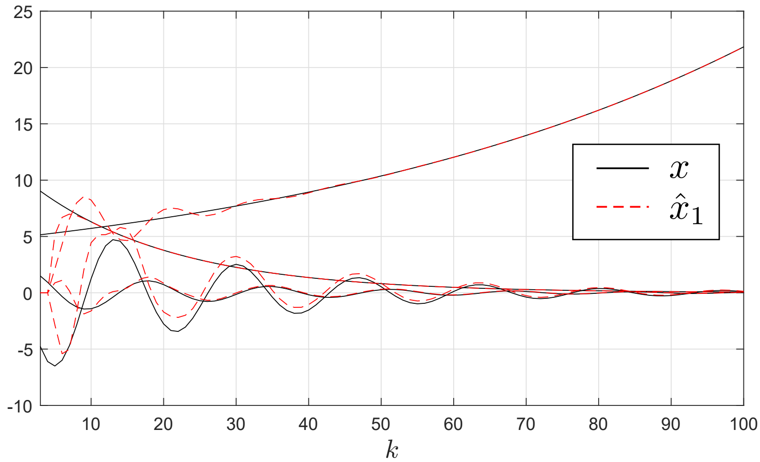

Figure 5.1: Evolution of the system state and agent 1 estimates (in dashed lines).

local information (and information exchanged in the distributed observer setup procedure) and that guarantees the stability of the distributed observer.

In the next chapter, a perturbed scenario of system dynamics and agents' measurements is considered. In this framework, a deign method that optimizes the noise rejection will be tackled.

Chapter 6

Data-fusion optimal filtering

6.1 Introduction

This chapter makes use of the data-fusion-based observer structure presented in (5.3). Remember that this structure is used to solve the distributed estimation problem relaying in the concept of data fusion. Data fusion is the process of integrating multiple data sources to produce more consistent, accurate, and useful information than that provided by any singular data source [24]. In this framework, each agent knows the model of the plant and can measure some system outputs affected by noise. The rest of the necessary outputs are obtained through the exchange of information with neighboring agents.

The objective of the chapter is to present a design method for the observer structure that minimizes the expected value of the norm of the estimation error. For this purpose, some assumptions are made. The solution is tested under simulation in order to show the robustness and effectiveness of the method.

6.2 Problem formulation

Consider a set of agents $\mathcal{V} = \{1, 2, \ldots, p\}$ connected according to a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and intended to distributedly estimate the state of the following discrete-time linear time-invariant system:

$$x(k+1) = Ax(k) + w(k), \tag{6.1}$$

$$y_i(k) = C_i x(k) + n_i(k), \tag{6.2}$$

where $w \in \mathbb{R}^n$ and $n_i \in \mathbb{R}^{m_i}$ are mutually independent Gaussian state and measurements noises, respectively, with covariance matrices $M \in \mathbb{R}^{n\times n}$ and

$R_i \in \mathbb{R}^{m_i \times m_i}$.

Note that, on the contrary to the approach in Chapter 4, the system and measurements perturbations are considered Gaussian with mean zero and covariance known.

As it was done in the previous chapters, the Assumption 2.3.4 for the network connectivity is required.

6.3 Observer structure and design goal

The observer considered in the further developments is the one first presented in Chapter 5 and whose structure is exposed next:

$$\begin{aligned} \hat{x}_i(k+1) &= A\hat{x}_i(k) \\ &+ \sum_{\rho=0}^{\ell_i} W_{i,\rho} N_{i,\rho}(k) W_{i,\rho}^\top \sum_{j \in \mathcal{N}_{i,\rho}} C_j^\top \left(y_j(k-\rho) - C_j \hat{x}_i(k-\rho)\right). \end{aligned} \tag{6.3}$$

The goal of this paper is to design the gain matrices $N_{i,\rho}(k)$ in structure (6.3) for every agent i and every hop $\rho \in \{0, \ldots, \ell_i\}$ to solve the following problem:

Problem 6.3.1 *(Distributed optimal filtering) Given plant* (6.1)-(6.2) *and the interconnection graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, design gains $N_{i,\rho}(k)$ in* (6.3) *in order to minimize the value of* $\mathbb{E}\{||\varepsilon_{i,\rho}(k+1)||^2\}$.

Please, consider in all the chapter the same distributed observer setup procedure than the one defined in Section 5.3.1.

6.4 Estimation error dynamics

This section presents the evolution of the transformed estimation error, $\varepsilon_{i,\rho}(k)$. The expression obtained will be similar to the one described in the previous chapter but, this time, perturbations are taken into consideration.

Proposition 6.4.1 *Consider the network of agents described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where every agent i implements the observer structure* (6.3) *to estimate the state of the system* (6.1). *Then, the transformed estimation error dynamics at every hop ρ is given by the following equation:*

$$\begin{aligned}\varepsilon_{i,\rho}(k+1) &= W_{i,\rho}^{\top} A \sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r}(k) + W_{i,\rho}^{\top} w(k) \\ &- N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top}\left(C_j \sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r}(k-\rho) + n_j(k-\rho)\right),\end{aligned} \tag{6.4}$$

for all $\rho = \{0,\ldots,\ell_i\}$.

Proof 6.4.1 *Let us write first the evolution of the estimation error dynamics for system* (6.1) *under the observation structure in* (6.3)*:*

$$\begin{aligned} e_i(k+1) = x(k+1) - \hat{x}_i(k+1) = Ae_i(k) \\ -\sum_{\rho=0}^{\ell_i} W_{i,\rho} N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top}\left(C_j e_i(k-\rho) + n_j(k-\rho)\right) + w(k).\end{aligned}$$

Using (2.10)*, we can write the transformed estimation error dynamics for agent* i *at hop* ρ*:*

$$\begin{aligned}\varepsilon_{i,\rho}(k+1) = W_{i,\rho}^{\top} e_i(k+1) = W_{i,\rho}^{\top} A e_i(k) \\ - N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top}\left(C_j e_i(k-\rho) + n_j(k-\rho)\right) + W_{i,\rho}^{\top} w(k),\end{aligned}$$

where $W_{i,\rho}^{\top} W_{i,\rho} = I_{n_{i,\rho}}$ *and Lemma 2.3.1 (i) has been used. Next, relying on Equation* (2.11)*, we know the expression of* e_i *in* $\varepsilon_{i,\rho}$ *coordinates, so previous equation can be accordingly modified:*

$$\begin{aligned}\varepsilon_{i,\rho}(k+1) &= W_{i,\rho}^{\top} A\left(\bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1}(k) + \sum_{r=0}^{\ell_i} W_{i,r}\varepsilon_{i,r}(k)\right) + W_{i,\rho}^{\top} w(k) \\ &- N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top}\left(C_j\left(\bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1}(k-\rho)\right.\right. \\ &+ \left.\left.\sum_{r=0}^{\ell_i} W_{i,r}\varepsilon_{i,r}(k-\rho)\right) + n_j(k-\rho)\right).\end{aligned}$$

From Lemma 2.3.1 (iv) it holds $W_{i,\rho}^{\top} A\left(\bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1} + \sum_{r=\rho+1}^{\ell_i} W_{i,r}\varepsilon_{i,r}\right) = 0$

so we can rewrite the above equation as:

$$\varepsilon_{i,\rho}(k+1) = W_{i,\rho}^{\top} A \sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r}(k) + W_{i,\rho}^{\top} w(k) - N_{i,\rho}(k) W_{i,\rho}^{\top}$$
$$\times \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top}\left(C_j \bar{V}_{i,\ell_i}\varepsilon_{i,\ell_i+1}(k-\rho) + C_j \sum_{r=0}^{\ell_i} W_{i,r}\varepsilon_{i,r}(k-\rho) + n_j(k-\rho)\right).$$

Next, from Lemma 2.3.2 we know that $C_j W_{i,\rho'} = 0$ *for all* $j \in \mathcal{N}_{i,\rho}$ *with* $\rho' > \rho$ *and therefore*

$$\begin{aligned}\varepsilon_{i,\rho}(k+1) &= W_{i,\rho}^{\top} A \sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r}(k) + W_{i,\rho}^{\top} w(k)\\ &- N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top}\left(C_j \sum_{r=0}^{\rho} W_{i,r}\varepsilon_{i,r}(k-\rho) + n_j(k-\rho)\right),\end{aligned}$$

which is the desired expression. □

Finally, note that Proposition 5.4.2 in Chapter 5, can be reformulated for the perturbed scenario considered as:

Proposition 6.4.2 *Consider the network of agents described by the graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, *where every agent* i *implements the observer structure* (6.3) *to estimate the state of the system* (6.1). *Then, the delayed transformed estimation error dynamics at hop* ρ *is given by the following equation:*

$$\begin{aligned}\bar{\varepsilon}_{i,\rho}(k+1) &= \sum_{r=0}^{\rho} \triangle_{i,(\rho,r)}(k)\bar{\varepsilon}_{i,r}(k) + S_{i,\rho}^{\top} W_{i,\rho}^{\top} w(k)\\ &- S_{i,\rho}^{\top} N_{i,\rho}(k) W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} n_j(k-\rho).\end{aligned} \tag{6.5}$$

The proof of this proposition comes straightforward from the proof of Proposition 5.4.2 but considering the perturbed system (6.1)-(6.2).

6.5 Distributed optimal filtering

This section deals with Problem 6.3.1, that is, gains $N_{i,\rho}(k)$ must be designed in such a way that the expected value of the norm of the delayed transformed estimation error of agent i at hop ρ, that is, $\mathbb{E}\{||\varepsilon_{i,\rho}(k+1)||^2\}$, is minimized. Note that

$\mathbb{E}\{||\varepsilon_{i,\rho}(k+1)||^2\} = \mathbb{E}\{\varepsilon_{i,\rho}^\top(k+1)\varepsilon_{i,\rho}(k+1)\} = \mathrm{tr}(\mathbb{E}\{\varepsilon_{i,\rho}(k+1)\varepsilon_{i,\rho}^\top(k+1)\}) = \mathrm{tr}(\mathbb{E}\{Q_{i,\rho}(k+1)\})$, where $Q_{i,\rho}$ is the covariance matrix of the transformed estimation error of agent i at hop ρ. The dynamics of the covariance matrix will be studied and, then, it will be possible to present a method to find the gains $N_{i,\rho}(k)$ to minimize $\mathrm{tr}(\mathbb{E}\{Q_{i,\rho}(k+1)\})$.

Let $\bar{Q}_{i,(\rho,r)}(k) = \mathbb{E}\{\bar{\varepsilon}_{i,\rho}(k)\bar{\varepsilon}_{i,r}^\top(k)\}$ be the cross covariance matrix of the delayed transformed estimation error of agent i between hops ρ and r. Thus, it is possible to conform the covariance matrix of the delayed transformed estimation error of agent i for every hop ρ:

$$
\begin{aligned}
\bar{Q}_i(k) &= \mathbb{E}\{\bar{\varepsilon}_i(k)\bar{\varepsilon}_i(k)^\top\} \\
&= \mathbb{E}\left\{\begin{bmatrix} \bar{\varepsilon}_{i,\ell_i}(k)\bar{\varepsilon}_{i,\ell_i}(k)^\top & \bar{\varepsilon}_{i,\ell_i}(k)\bar{\varepsilon}_{i,\ell_i-1}(k)^\top & \cdots & \bar{\varepsilon}_{i,\ell_i}(k)\bar{\varepsilon}_{i,0}(k)^\top \\ \bar{\varepsilon}_{i,\ell_i-1}(k)\bar{\varepsilon}_{i,\ell_i}(k)^\top & \bar{\varepsilon}_{i,\ell_i-1}(k)\bar{\varepsilon}_{i,\ell_i-1}(k)^\top & \cdots & \bar{\varepsilon}_{i,\ell_i-1}(k)\bar{\varepsilon}_{i,0}(k)^\top \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\varepsilon}_{i,0}(k)\bar{\varepsilon}_{i,\ell_i}(k)^\top & \bar{\varepsilon}_{i,0}(k)\bar{\varepsilon}_{i,\ell_i-1}(k)^\top & \cdots & \bar{\varepsilon}_{i,0}(k)\bar{\varepsilon}_{i,0}(k)^\top \end{bmatrix}\right\} \\
&= \begin{bmatrix} \bar{Q}_{i,(\ell_i,\ell_i-1)}(k) & \bar{Q}_{i,(\ell_i,\ell_i)}(k) & \cdots & \bar{Q}_{i,(\ell_i,0)}(k) \\ \bar{Q}_{i,(\ell_i-1,\ell_i-1)}(k) & \bar{Q}_{i,(\ell_i-1,\ell_i)}(k) & \cdots & \bar{Q}_{i,(\ell_i-1,0)}(k) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_{i,(0,\ell_i-1)}(k) & \bar{Q}_{i,(0,\ell_i)}(k) & \cdots & \bar{Q}_{i,(0,0)}(k) \end{bmatrix}.
\end{aligned}
$$

It is worth pointing out that it is possible to relate the diagonal terms of the covariance matrix $\bar{Q}_i(k)$ through the use of the selection matrices defined in Section 5.4. Therefore, $Q_{i,\rho}(k) = S_{i,\rho}\bar{Q}_{i,(\rho,\rho)}(k)S_{i,\rho}^\top = \bar{S}_{i,\rho}\bar{Q}_i(k)\bar{S}_{i,\rho}^\top$.

Note that matrices $\triangle_{i,(\rho,r)}$ and $\triangle_{i,(\rho,\rho)}$ in (5.8) and (5.9) for $r,\rho \in \{0,\ldots,\ell_i\}$ can be rewritten as:

$$
\triangle_{i,(\rho,r)}(k) = \tilde{\triangle}_{i,(\rho,r)} - S_{i,\rho}^\top N_{i,\rho}(k)F_{i,(\rho,r)},
$$

where $F_{i,(\rho,r)} = \begin{bmatrix} 0 & 0 & \cdots & W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top C_j W_{i,r} & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n_{i,\rho}\times(\ell_i+1)n_{i,r}}$ and

$$
\tilde{\triangle}_{i,(\rho,\rho)} = \begin{bmatrix} W_{i,\rho}^\top A W_{i,\rho} & 0 & \cdots & 0 & 0 \\ I_{n_{i,\rho}} & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & \cdots & I_{n_{i,\rho}} & 0 \end{bmatrix},
$$

$$\tilde{\triangle}_{i,(\rho,r)} = \begin{bmatrix} W_{i,\rho}^{\top} A W_{i,r} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

for every $r, \rho \in \{0, \dots, \ell_i\}$ with $r \neq \rho$. Using this decomposition, the next proposition studies the dynamics of the complete covariance matrix $\bar{Q}_i$. Its proof requires just some simple mathematical manipulations and, hence, it is omitted.

Proposition 6.5.1 *The evolution of covariance matrix $\bar{Q}_i$ is given by:*

$$\begin{aligned} \bar{Q}_i(k+1) &= \left(\tilde{\triangle}_i - S_i^{\top} N_i(k) F_i\right) \bar{Q}_i(k) \left(\tilde{\triangle}_i - S_i^{\top} N_i(k) F_i\right)^{\top} \\ &+ S_i^{\top} V_{i,\ell_i}^{\top} M V_{i,\ell_i} S_i - S_i^{\top} N_i(k) D_i N_i^{\top}(k) S_i, \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} N_i(k) &= diag\left(N_{i,\rho}(k)\right)_{\rho\in\{\ell_i,\dots,0\}}, \\ D_i &= diag\left(W_{i,\rho} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} R_j C_j W_{i,\rho}^{\top}\right)_{\rho\in\{\ell_i,\dots,0\}}, \end{aligned}$$

and:

$$\tilde{\triangle}_i = \begin{bmatrix} \tilde{\triangle}_{i,(\ell_i,\ell_i)} & \tilde{\triangle}_{i,(\ell_i,\ell_i-1)} & \cdots & \tilde{\triangle}_{i,(\ell_i,0)} \\ 0 & \tilde{\triangle}_{i,(\ell_i-1,\ell_i-1)} & \cdots & \tilde{\triangle}_{i,(\ell_i-1,0)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\triangle}_{i,(0,0)} \end{bmatrix}, \; S_i = \begin{bmatrix} \bar{S}_{i,\ell_i} \\ \bar{S}_{i,\ell_i-1} \\ \vdots \\ \bar{S}_{i,0} \end{bmatrix},$$

$$F_i = \begin{bmatrix} F_{i,(\ell_i,\ell_i)} & F_{i,(\ell_i,\ell_i-1)} & \cdots & F_{i,(\ell_i,0)} \\ 0 & F_{i,(\ell_i-1,\ell_i-1)} & \cdots & F_{i,(\ell_i-1,0)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_{i,(0,0)} \end{bmatrix},$$

Based on this proposition we present a design method for gains $N_{i,\rho}(k)$ that solves the Problem 6.3.1.

Theorem 6.5.1 *The set of matrices $N_{i,\rho}(k)$ that minimize tr $(Q_{i,\rho}(k+1))$ are the ones that solve the following system of equations:*

$$N_{i,\rho}(k)\left(W_{i,\rho}^{\top} \sum_{j\in\mathcal{N}_{i,\rho}} C_j^{\top} R_j C_j W_{i,\rho} - F_{i,(\rho,:)} \bar{Q}_i(k) F_{i,(\rho,:)}^{\top}\right) = \bar{S}_{i,\rho} \tilde{\triangle}_i \bar{Q}_i(k) F_{i,(\rho,:)}^{\top}, \tag{6.7}$$

for every $\rho \in \{0, \ldots, \ell_i\}$, *where* $F_{i,(\rho,:)} = [0 \ \cdots \ F_{i,(\rho,\rho)} \ \cdots \ F_{i,(\rho,0)}]$ *denotes the row block* $\ell_i - \ \rho + 1$ *of matrix* F_i.

Proof 6.5.1 *The proof is divided into several steps. Firstly, the dynamics of matrix* $Q_{i,\rho}$ *will be obtained derived from results in Proposition 6.5.1. Then, its trace will be derived with respect to* $N_{i,\rho}(k)$ *in order to find the value of the gain that minimizes the trace of* $Q_{i,\rho}(k)$.

Observe that $Q_{i,\rho}(k) = \bar{S}_{i,\rho}\bar{Q}_i(k)\bar{S}_{i,\rho}^\top$. *Then, it holds:*

$$\begin{aligned} Q_{i,\rho}(k+1) &= \left(\bar{S}_{i,\rho}\tilde{\triangle}_i - \bar{S}_{i,\rho}S_i^\top N_i(k)F_i\right)\bar{Q}_i(k)\left(\bar{S}_{i,\rho}\tilde{\triangle}_i - \bar{S}_{i,\rho}S_i^\top N_i(k)F_i\right)^\top \\ &+ \ \bar{S}_{i,\rho}S_i^\top V_{i,\ell_i}^\top M V_{i,\ell_i} S_i \bar{S}_{i,\rho}^\top - \bar{S}_{i,\rho}S_i^\top N_i(k) D_i N_i^\top(k) S_i \bar{S}_{i,\rho}^\top. \end{aligned}$$

It is a simple matter to check that $\bar{S}_{i,\rho}S_i^\top V_{i,\ell_i}^\top = W_{i,\rho}^\top$ *and* $\bar{S}_{i,\rho}S_i^\top N_i(k)F_i = N_{i,\rho}(k)F_{i,(\rho,:)}$, *and, consequently, we can rewrite the above expression as:*

$$\begin{aligned} Q_{i,\rho}(k+1) &= \left(\bar{S}_{i,\rho}\tilde{\triangle}_i - N_{i,\rho}(k)F_{i,(\rho,:)}\right)\bar{Q}_i(k)\left(\bar{S}_{i,\rho}\tilde{\triangle}_i - N_{i,\rho}(k)F_{i,(\rho,:)}\right)^\top \\ &+ \ W_{i,\rho}^\top M W_{i,\rho} - N_{i,\rho}(k)W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}(k)} C_j^\top R_j C_j W_{i,\rho} N_{i,\rho}^\top(k). \end{aligned}$$

Now, with the purpose of minimizing the trace of the covariance matrix, $tr(Q_{i,\rho}(k+1))$ *is partially derived with respect to the gain matrix* $N_{i,\rho}(k)$*:*

$$\begin{aligned} \frac{\partial tr(Q_{i,\rho}(k+1))}{\partial N_{i,\rho}(k)} &= -2\bar{S}_{i,\rho}\tilde{\triangle}_i\bar{Q}_i(k)F_{i,(\rho,:)}^\top + 2N_{i,\rho}(k)F_{i,(\rho,:)}\bar{Q}_i(k)F_{i,(\rho,:)}^\top \\ &- \ 2N_{i,\rho}(k)W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top R_j C_j W_{i,\rho}. \end{aligned}$$

Then, equaling the above expression to zero, we can find that the gain $N_{i,\rho}(k)$ *that minimizes the trace of* $Q_{i,\rho}(k+1)$ *satisfies*

$$\begin{aligned} &\bar{S}_{i,\rho}\tilde{\triangle}_i\bar{Q}_i(k)F_{i,(\rho,:)}^\top + N_{i,\rho}(k)F_{i,(\rho,:)}\bar{Q}_i(k)F_{i,(\rho,:)}^\top \\ &\qquad = N_{i,\rho}(k)W_{i,\rho}^\top \sum_{j\in\mathcal{N}_{i,\rho}} C_j^\top R_j C_j W_{i,\rho}. \qquad (6.8) \end{aligned}$$

After some manipulations, expression (6.7) *is found.* $\square$.

This theorem has proposed a design method for gain matrices $N_{i,\rho}(k)$ that minimize the expected value of the estimation error. Additionally, the design method only requires solving a system of linear equations that depends only on local information of the agent, allowing its resolution in a distributed way.

The algorithm that every agent $i \in \mathcal{V}$ must follow to initialize the distributed estimation procedure and run the estimation phase is summarized in Table 6.1.

Table 6.1: Iterative algorithm.

Algorithm.	
1:	**Initialization:**
2:	Run the Distributed observer setup algorithm exposed in Section 5.3.1.
3:	Initialize covariance matrix $\bar{Q}_{i,\rho}$.
4:	**Iterative update:**
5:	While (1):
6:	Obtain $N_{i,\rho}(k)$ from Theorem 6.5.1 for every $\rho \in \{0, \dots, \ell_i\}$.
7:	Obtain $\bar{Q}_i(k+1)$ from Proposition 6.5.1.
8:	Exchange the corresponding information with the neighborhood.
9:	Obtain $\hat{x}_i(k+1)$ from Equation (5.3).
10:	End while

6.6 Simulation results

In order to show the effectiveness and optimality of the distributed observer some simulations are driven in this section. Consider the following system where there is one state with a stable dynamics, a pair of conjugated imaginary poles and a state with an unstable dynamics:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 & 0 \\ 0 & 0.8606 & -1.3368 & 0 \\ 0 & 0.0941 & 0.9315 & 0 \\ 0 & 0 & 0 & 1.015 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}. \tag{6.9}$$

The system is being observed by a set of three agents with the network topology depicted in Figure 6.1.

Example 6.6.1 *In this example the performance of the observer is tested under the perturbed scenario in which the system model and the agents measurements are affected*

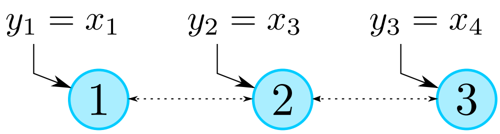

Figure 6.1: Network topology considered.

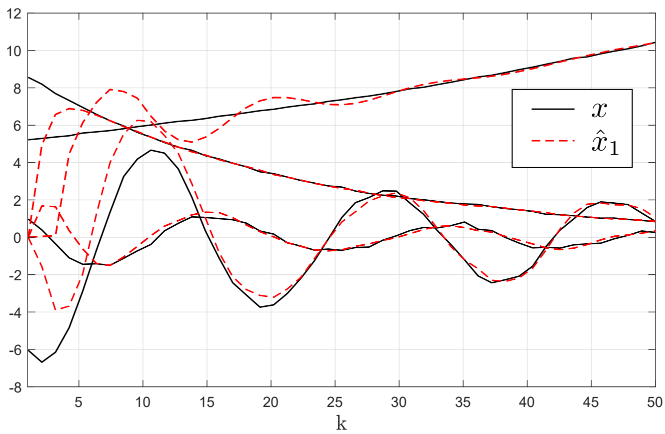

Figure 6.2: Evolution of the system state and agent 1 estimates (in dashed lines).

by Gaussian noises. Consider the sequel covariance matrices of the noises terms:

$$M = \begin{bmatrix} 0.02 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & 0.06 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}, \quad R_1 = 0.01, \quad R_2 = 0.02, \quad R_3 = 0.01.$$

Figure 6.3 depicts the evolution of the system state and agent 1 estimates (in dashed lines). It is shown that the estimate of the stable pole that is locally measured by agent 1 converges the first to the real value. After it, the estimates of the states 2 and 3, measured by agent 2, reach the actual value. Finally, the estimator converges to the values of the unstable pole, measured by agent 3 (that belongs to $\mathcal{N}_{i,2}$). This simple example makes it clear that the decay rate of the estimator error decreases with ρ. In Figure 6.2 the evolution of $tr(Q_{i,\rho})$ for every $i \in \mathcal{V}$ are shown.

To conclude this section, it is shown that the gain matrices obtained in this example in steady state (through the use of Theorem 6.5.1), fulfill the LMI stated in Theorem 5.5.2 and, consequently, meet the stability conditions required in Theorem 5.5.1. Considering for example agent 1 with the gain matrices $N_{1,\rho}$

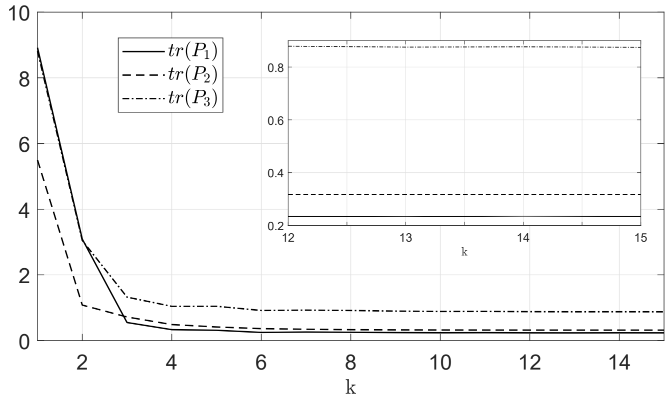

Figure 6.3: Evolution of $tr(Q_{i,\rho})$ for every agent.

obtained through Theorem 6.5.1, it is possible to find a solution for the LMI (5.11) on the variables $\mathcal{X}_{1,\rho}$ and $\mathcal{Y}_{1,\rho}$, and therefore to guarantee the stability of the estimation errors. The values obtained for $N_{1,\rho}$, $\mathcal{X}_{i,\rho}$, and $\mathcal{Y}_{1,\rho}$ are given in Table 6.2.

Table 6.2: Values of $\mathcal{X}_{1,\rho}$, $\mathcal{Y}_{1,\rho}$ and $N_{1,\rho}$ obtained in Theorem 6.5.1 for agent 1.

	Agent 1	
	$\mathcal{X}_{1,\rho}$	$\mathcal{Y}_{1,\rho}$
$\rho = 0$	64.846	$\begin{bmatrix} 57.643 & 0 \\ 0 & 41.431 \end{bmatrix}$
$\rho = 1$	$\begin{bmatrix} 0.801 & 0.218 \\ 0.218 & 1.090 \end{bmatrix}$	$\begin{bmatrix} 2.165 & 0.435 & 0.671 & -0.033 \\ 0.435 & 2.472 & -0.289 & -0.002 \\ 0.671 & -0.289 & 2.405 & 0.075 \\ -0.033 & -0.002 & 0.075 & 2.909 \end{bmatrix}$
$\rho = 2$	38.425	$\begin{bmatrix} 44.862 & 0 \\ 0 & 51.814 \end{bmatrix}$
	$N_{1,0} = 0.577, \quad N_{1,1} = \begin{bmatrix} 0.363 & 0 \\ -1.274 & 0 \end{bmatrix}, \quad N_{1,2} = 0.418$	

6.7 Conclusions

In this chapter, we have presented a design method to the data-fusion-based observer strcture presented previously. The design guarantees the stability of the observer and ensures the minimization of the expected value of the estimation error norm when the perturbations considered are uncorrelated and follow normal distributions with mean zero and known covariance.

The design is tested under simulation and the results obtained are related with the stability proofs stated in Chapter 5.

Chapter 7

Conclusions

7.1 Main achievements

This thesis has been focused on solving the distributed estimation problem defined in Section 1.4. In particular, the thesis has proposed four methods for solving the problem under necessary and sufficient conditions in terms of agents connectivity and system and agents detectability.

The main contributions can be divided into four different areas according to the main chapters (Chapters 3, 4, 5 and 6). All of them are based on solving the problem through the use of a state space transformation that is formally defined in Chapter 2. This form allows each agent involved in the network to identify its observable modes at each hop ρ; that is, by using this transformation, it is possible to identify the observable modes accessible by the agent with local information and the information provided by those agents with a direct path to it, involving ρ or less edges. The contributions are detailed below:

- Chapter 2
    1. A multi-hop subspace decomposition has been presented. In this decomposition, the state space is divided according to the modes detectable by each agent counting with the information provided by its neighborhood at different hops.
    2. The decomposition presented allows to transform the system dynamics into an upper block triangular matrix revealing a cascade structure.
    3. It has been shown that, contrary to most of the approaches found in literature, only necessary assumptions are required in terms of agents connectivity.

- Chapter 3

  1. This chapter presented a novel observer structure that makes use of a subspace decomposition. Based on that transformation, it is proven that the observer can be designed in a distributed manner, only requiring the necessary exchange of information in a distributed observer set-up phase.
  2. A theoretical proof is provided, showing that it is always possible to find, under necessary and sufficient assumptions, a set of observer gains that stabilizes the estimation.
  3. The observer structure presented reduces the exchange of information during the running phase, i.e., after the observer gains have been designed. This result is due to the fact that, relaying in the subspace decomposition, the neighboring agents only need to send to each agent the part of the state not observable by itself.
  4. The developed design method is computationally simple. It can be carried out through the use of a simple pole placement algorithm.
  5. An arbitrary convergence rate can be fixed for each agent.

- Chapter 4

  1. An observer LQ-based design has been introduced in the context of distributed estimation problems.
  2. The design proposed guarantees the stability of the observer under bounded disturbances.
  3. The observer design can be simplified to tune just one scalar parameter that trades off between a fast convergence rate and good noise rejection. This parameter can be tuned by a control engineer based on their experience of the process.

- Chapter 5

  1. This last chapter presented a data-fusion-based algorithm to solve the problem of distributed estimation of the state of a plant. This algorithm also makes use of a observer structure based on the subspace decomposition presented in the thesis.
  2. The agents involved in the network can conveniently modify the construction of the transformation matrix, which allows one to perform

subspace decomposition to be selective with the agents that act as a source of information.

3. Unlike the conventional data fusion approaches, the information is not required to be spread through the network in a single sample time, thereby relaxing the requirements of the network.
4. The exchange of information required to implement the algorithm is less than other data-fusion-based approaches, lowering the cost by considering a lower amount of redundant information.

- Chapter 6

  1. A data-fusion-based observer design has been introduced in the context of distributed estimation problems.
  2. The proposed design minimizes the expected value of the estimation error norm when the perturbations are considered Gaussian with mean zero and covariance known.

7.2 Potential weakness and limitations

The previous section described the achievements and potential advantages of the distributed observer structures introduced in the thesis. However, it is important to note that a preliminary study of the system and the interconnected network of agents has to be previously undertaken to implement the proposed distributed observer. This is crucial due to the fact that the estimation structure presented cannot be the most suitable approach for all the cases.

7.2.1 Reliable communication networks

The subspace decomposition used allows the observer to be selective with the source of information considered for reconstructing the state. This feature is important when working with slow networks in which the minimization of the information exchange is crucial or when working with information highly perturbed by noise or cyberattacks. However, there are some scenarios under which this feature can be a drawback.

Let us consider the case in which we want to implement a distributed algorithm to estimate the state of a perturbed system. Consider also that the measurement taken by the agents is highly affected by noise and that the agents' interconnected network is not limited, i.e. it has not limitation in terms of volume of information and transmission rate. In this framework, being selective with the information used by each agent to reconstruct the state is not useful. A data fusion algorithm considering the measurements taken by all agents will result in better performance in estimation due to the fact that better filtering can be achieved by considering redundant information on the system state.

On the other hand, the distributed observer setup procedure allows one to select the agents that act as a source of information. This selection is carried out according to the number of hops from the source to the target agent. This criterion does not have to be the most optimal in relation to the measurements and the dynamics of the system.

7.2.2 System dynamics

The examples presented in the simulations of Chapters 2, 3 and 4 have shown the performance of the proposed algorithms. The system matrices have been selected for the particular purpose of showing a clarifying example. Nevertheless, when modeling a system's dynamics, it is common to encounter relations among every state considered. In this sense, most of the relations are weak and almost zero. However, this fact must be taken into account to accomplish the subspace decomposition.

For instance, let us suppose the following system and output matrices:

$$A = \begin{bmatrix} 100 & 0.1 \\ 0.1 & 100 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Consider also that both agents have a communication link between them. As it is shown, both pairs (A, C_1) and (A, C_2) are observables. However, if the state is reconstructed in each agent based only on local measurements, the convergence rate for the estimation of the state with a "weak connection" will be too slow. Additionally, if the evolution of the system or the measurements taken are affected by noise, the amplitude of this noise can be greater than the relation established by the system dynamics, which makes it impossible to estimate that portion of the state.

In this situation, it is important to consider the weak links in the system dynamics as "zeros". Thus, the subspace decomposition will work properly and achieve optimal performance.

7.3 Further work

This thesis has proven the potential of the state space decomposition presented for the problem of distributed estimation. Some extensions of this initial idea have been shown in Chapters 3 and 4, where the consideration of noise has been taken into account. Hence, based on this estimator, some further developments of the work are possible:

- Study the observer structure behavior dealing with random events, specifically considering that node neighborhoods can change randomly as a result of random link dropouts and recovery.
- Develop new algorithms to face situations in which agents can receive false data from their neighbors due to the fact that some agents can become faulty or under the control of non-authorized entities.
- Explore the possibility of finding an observer structure that only decouples the state space in the observable and unobservable subspace for each agent. Thus, the observable modes are reconstructed only based on local measurement of the state, and the unobservable modes are obtained by consensus with the rest of the network.
- Analyze the robustness of the observer dealing with delays and data dropouts.

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